Negatively Curved Graph and Planar Metrics with Applications to Type

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Introduction

A graph is of parabolic or hyperbolic type if the simple random walk on the vertices is, respectively, recurrent or transient. A plane triangulation graph is CP-parabolic or CP-hyperbolic if the maximal circle packing determined by the graph packs, respectively, the complex plane $\mathbb{C}$ or the Poincaré disk $\mathbb{D}$. We examine the implications that (Gromov) negative curvature carries for determining type, specifically in these settings. Our main result is encased in the following theorem.

Theorem. *Every proper* (*Gromov*) *negatively curved metric space whose boundary contains a nontrivial continuum admits a* $(2, C)$*-quasi-isometric embedding of a uniform binary tree.*

Corollaries of this theorem include:

(1) the simple random walk on every locally finite, negatively curved graph whose boundary contains a nontrivial continuum is transient;
(2) the simple random walk on a locally finite, 1-ended negatively curved graph whose boundary contains more than one point is transient;
(3) a negatively curved plane triangulation graph is CP-hyperbolic if and only if it has a circle boundary (equivalently, CP-parabolic if and only if it has a point boundary).

The classical "type problem" is that of determining whether a given noncompact, simply connected Riemann surface is conformally equivalent to the plane $\mathbb{C}$ or the disk $\mathbb{D}$. The surface is said to be of *parabolic* type in the former case, and of *hyperbolic* type in the latter. Our concern is with two related discretizations of this classical problem, one via random walks on graphs, the other via planar circle packings. Connections between probabilistic characteristics and the type problem are deep and intimate, and have been known for a long time. For instance, a simply connected Riemann surface is hyperbolic if and only if a Brownian traveler starting at any point has a positive escape probability. This generalizes to

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