Wandering Property in the Hardy Space

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1. Introduction

Let X be a Hilbert space and let $V: X \to X$ be a bounded linear operator. If V is an isometry, then the well-known Wold decomposition theorem states that

$$X = X_0 \bigoplus_{n=0}^{\infty} V^n X_1, \tag{1}$$

where $X_1 = X \ominus VX$ is a wandering subspace and $X_0 = \bigcap_{n=0}^{\infty} V^n X$ [4]. If $X = H^2$ and V is the operator of multiplication by an inner function g, then the intersection $\bigcap_{n=0}^{\infty} V^n H^2 = \{0\}$ and the decomposition (1) implies that an orthonormal basis of $H^2 \ominus gH^2$, $\{s_1, \dots, s_n, \dots\}$, is a g-basis of H^2; that is, any function $f \in H^2$ can be written as

$$f(z) = \sum_{n=0}^{\infty} s_n(z) f_n(g(z)). \tag{2}$$

Any closed subspace $M \subset H^2$ that is invariant under multiplication by g could be considered as X, and therefore a relation similiar to (2) holds. We write this relation in the following form. Given a subset $A \subset H^2$, we denote by $[A]_g$ the minimal closed subspace of H^2 containing A that is invariant under multiplication by g. In our case the relation (1) could be written in these terms as follows. If M is invariant under multiplication by g, then

$$[M \ominus gM]_g = M. \tag{3}$$

It was shown in [5] that the relation (3) yields in this case some nice properties of functions from $M \ominus gM$ and leads to a generalization of classical canonical factorization. In general, (3) leads to description of multiplication invariant subspaces. In the case of the Bergman space in the unit disk, the validity of (3) when $g(z) = z$ was proved in [1].

In this paper we investigate the question when (3) holds if g is not inner. More precisely, let g be a bounded analytic function in the unit disk. We ask when (3) holds for any subspace $M \subset H^2$ that is invariant under multiplication by g. Our main result is the following.

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