Pluricomplex Green Functions and the Dirichlet Problem for the Complex Monge–Ampère Operator

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0. Preliminaries

Let us recall some important notions which will be used here. Let D be an open subset in $\mathbb{C}^n$, and denote by PSH(D) the cone of plurisubharmonic functions $u: D \to [-\infty, +\infty[$ on D not identically equal to $-\infty$ on any component of D.

Let $u \in \text{PSH}(D)$. For $a \in D$ and $0 < r < d_a := \text{dist}(a; \mathbb{C}^n \setminus D)$, we set

$$M_u(a, r) := \int_{|\xi|=1} u(a + r\xi)\, d\sigma(\xi), \tag{0.1}$$

where $d\sigma(\xi)$ is the normalized area measure on the unit Euclidean sphere in $\mathbb{C}^n$. It is well known that the function $r \mapsto M_u(a, r)$ is increasing and convex in $\log r$. Then the following limit exists:

$$\nu(u; a) := \lim_{r \to 0^+} \frac{M_u(a, r)}{\log r}. \tag{0.2}$$

By [Ki1], (0.2) coincides with the following definition [L1]:

$$\nu(u; a) := \lim_{r \to 0^+} \frac{\sigma_u(B(a, r))}{\omega_{2n-2} r^{2n-2}}, \tag{0.3}$$

where ω_{2n-2} is the volume of the unit ball in $\mathbb{C}^{n-1}$ and $\sigma_u := \frac{1}{2\pi} \Delta u \beta_n = \frac{1}{2\pi} dd^c u \wedge \beta_{n-1}$; β is the standard Kälherian form of $\mathbb{C}^n$ and $\beta_{n-1} := \beta^{n-1}/(n-1)!$.

The number defined by (0.3) is called the *Lelong number* of the current $dd^c u$ at the point a, or the *density* of u at the point a. It is well known that the Lelong number is independent of holomorphic changes of coordinates [S; D3]. Thus it is possible to define this number for plurisubharmonic functions on complex manifolds. In fact, the definition (0.3) is meaningful in this case.

The function $\nu(u; \cdot): a \mapsto \nu(u; a)$ defined by (0.3) is upper semicontinuous on D, with values in $\mathbb{R}_+$. If $u(a) > -\infty$ then $\nu(u; a) = 0$. If $u = \log|f|$, where f is a holomorphic function such that $f(a) = 0$ and not identically zero on a neighborhood of a, then $\nu(\log|f|; a)$ is an integer equal to the multiplicity of the zero of f at the point a.

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