The Minimal Norm Property for Quadratic Differentials in the Disk

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1. Introduction

Let Δ be the unit disk. We let $M(\Delta)$ be the open unit ball of $L^\infty(\Delta)$. For any μ in $M(\Delta)$, there exists a solution $f\colon \Delta \to \Delta$ of the Beltrami equation

$$f_{\bar{z}} = \mu f_z, \tag{1}$$

unique up to a postcomposition by a Möbius transformation. We call f a *quasiconformal* homeomorphism of a disk with the Beltrami coefficient μ, and we denote by f^μ the solution f of (1) normalized by $f(i) = i$, $f(1) = 1$, and $f(-1) = -1$. The solution f can be extended to a homeomorphism of the closure of Δ, and the restriction h of that extension to the boundary of Δ is called a *quasisymmetric* homeomorphism of a circle. The dilatation $K(h)$ of a quasisymmetric homeomorphism h is the infimum of all maximal dilatations of quasiconformal extensions of h to Δ. The boundary dilatation $H(h)$ of a quasisymmetric homeomorphism h is obtained by looking at the infimum of all maximal dilatations of quasiconformal extensions of h to a neighborhood U of the boundary and taking the limit of these dilatations as U shrinks to the boundary. We call a quasisymmetric homeomorphism h *symmetric* if $H(h) = 1$.

Let QC(Δ) be the space of all quasiconformal homeomorphisms of Δ. Two elements f_1, f_2 in QC(Δ) are *equivalent* if there exists a conformal homeomorphism α of Δ such that $f_1(t) = \alpha \circ f_2(t)$ for every $t \in \partial\Delta$. The Teichmüller space $T(\Delta)$ is QC(Δ) factored by this equivalence relation. The equivalence class of the identity mapping is called the *basepoint* of $T(\Delta)$.

We let $A(\Delta)$ be the Banach space of all holomorphic quadratic differentials φ on Δ satisfying $\|\varphi\| = \iint_\Delta |\varphi| < \infty$. One useful property of the Banach space $A(\Delta)$ is the following lemma, due to Strebel (see [S2]).

LEMMA 1. *Let φ be an arbitrary holomorphic quadratic differential of norm $\|\varphi\| \le M < \infty$ in the unit disk Δ. Let w be a boundary point of Δ. Then, for any $\varepsilon > 0$ and $\rho_2 > 0$, there exists a number ρ_1, $0 < \rho_1 < \rho_2$, such that*

$$\int_{\sigma_\rho} |\varphi(z)|^{1/2} |dz| < \varepsilon$$

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