Minimal Norm Interpolation with Nonnegative Real Part on Multiply Connected Planar Domains

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1. Introduction

Let Ω be a domain in the plane whose boundary is composed of a finite number of disjoint smooth simple closed curves. The space $H^2(\Omega)$ consists of those analytic functions f on Ω for which the subharmonic function $|f(z)|^2$ has a harmonic majorant. $K(\Omega)$ is the convex cone of those elements in $H^2(\Omega)$ whose real part is nonnegative on Ω.

In this paper we describe the projection of $H^2(\Omega)$ onto $K(\Omega)$ and also describe the unique element of $K(\Omega)$ of minimal norm satisfying a finite number of interpolation conditions:

$$\min\{\, \|f\|_{H^2(\Omega)} : f \in K(\Omega) \text{ and } f(z_j) = w_j,\ j = 1, 2, \dots, n\,\}, \tag{1.1}$$

assuming, of course, that there is at least one element of $K(\Omega)$ satisfying these conditions.

A problem similar to (1.1) was solved by Sarason [11] for the space $H^\infty(\Delta)$ where Δ is the open unit disc. He proved that the minimal norm interpolant is rational. As explained in [3], this result has importance in signal processing. The $H^2(\Omega)$ version of the problem studied here was suggested to us by J. D. Ward, to whom we express our appreciation.

The plan of the paper is this. In Sections 2 and 3 we give a description of the projection of $H^2(\Omega)$ onto $K(\Omega)$ in the case when Ω is a finitely connected domain. The special case when $\Omega = \Delta$ is examined separately in Section 2 because of its importance and simplicity. In Section 4 we show how this knowledge, combined with a result from [9], leads us to the the solution of problem (1.1) when $\Omega = \Delta$. In Section 5 we give some results similar to those of Section 3 but for finitely connected domains, where additional attention is given to the special case of an annulus.

2. The Projection onto Functions with Nonnegative Real Part: The Unit Disc

Let $\Delta = \{\, z : |z| < 1\,\}$ be the open unit disc, T the unit circle, and $dm = \frac{1}{2\pi}\, d\theta$ Lebesgue measure on T normalized so that $\int_T dm = 1$. Also, we use $\|\cdot\|_2$ to denote the L^2 norm relative to this measure.

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