# Polynomial Hulls of Sets in $\mathbb{C}^3$ Fibered over the Unit Circle

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## Introduction

Let $X$ be a compact subset of $\mathbb{C}^2$ lying over the unit circle $\mathbb{T}$; that is, letting $\pi(z, w) = z$ be the projection to the first coordinate, $\pi(z) \in \mathbb{T}$ for all $z \in X$. The fiber $X_z = \{ w \in \mathbb{C} : (z, w) \in X \}$ is identified with $\{z\} \times X_z = X \cap \pi^{-1}(z) \subseteq \mathbb{C}^2$. A number of authors [AW; S1; F] have discussed the polynomial hull $\hat{X}$ of $X$. The following definitive result was obtained by Slodkowski [S2].

THEOREM. *Suppose that each fiber $X_z$ is connected and simply connected. Then $\hat{X} \setminus X$ is the union of graphs of $H^\infty$ functions $h$ whose boundary values $h^*(z)$ are contained in $X_z$ for almost all $z \in \mathbb{T}$.*

Without the assumption that the fibers are connected, the conclusion no longer holds and in fact the hull may contain no analytic structure. This is the case in the example of Wermer [W] discussed below, where the fibers $X_z$ are totally disconnected.

The $H^\infty$ functions $h$ of the theorem are sometimes referred to as analytic selection functions. In the context of a connected compact plane set, "simply connected" means having connected complement in $\mathbb{C}$; this is equivalent to being polynomially convex. When $X$ is a compact subset of $\mathbb{C}^3$ lying over the unit circle, a statement like that of the above theorem—that the hull of $X$ is composed of the graphs of ($\mathbb{C}^2$-valued) selection functions—is no longer true, at least when the fibers $X_z$ are not linearly convex: this special case carries over to all dimensions [AW, S1]. Helton and Merino [HM] and Černe [C] have given examples of sets $X$ in $\mathbb{C}^3$ with "nice" fibers $X_z$ such that $\pi(\hat{X})$ is the closed unit disk but where no analytic selection functions exist. In their examples the reason that $\hat{X}$ covers the disk is that there is a 1-variety with boundary in $X$. In particular, $\hat{X}$ contains analytic structure. In general, however, as was first shown by Stolzenberg [St], polynomial hulls need not contain analytic structure. The purpose of this note is to construct a compact subset $X$ of $\mathbb{C}^3$, lying over the unit circle, such that $\hat{X}$ covers the unit disk but contains no analytic structure and such that the fibers are topologically simple, like those in Slodkowski's theorem—in our case the polynomially convex fibers $X_z \subseteq \mathbb{C}^2$ will be contractible to a point in $\mathbb{C}^2$. It is of interest to note

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