The Real Part of Entire Functions

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1. Introduction

Given an entire function

$$f(z) = \sum a_n z^n = u + iv,$$

let us write, as usual, $M(r)$ for the maximum modulus of f, and $A(r)$ and $B(r)$ for the minimum and maximum of u, the real part of f. We always have

$$-M(r) \le A(r) \le B(r) \le M(r),$$

but in fact the outer inequalities are, for most values of r, almost equalities. Wiman [14] showed that

$$-A(r) \sim B(r) \sim M(r)$$

as $r \to \infty$ outside an exceptional set of finite logarithmic measure, that is, outside a set E such that

$$\operatorname{logmeas} E = \int_{E\cap(1,\infty)} d\log t < \infty.$$

Hayman [10] obtained refinements of these estimates at the expense of a larger exceptional set, measured in terms of upper logarithmic density. The *upper* and *lower logarithmic densities* of E are defined by

$$\overline{\operatorname{logdens}}\, E = \overline{\lim_{r\to\infty}} \frac{\operatorname{logmeas} E_{(1,r)}}{\log r}, \qquad \underline{\operatorname{logdens}}\, E = \underline{\lim_{r\to\infty}} \frac{\operatorname{logmeas} E_{(1,r)}}{\log r},$$

where $E_{(1,r)}$ denotes the part of E contained in the interval $(1, r)$. Upper and lower $\log\log$ *densities* also arise in what follows and are defined analogously. Hayman proved the following.

Theorem 1 [10, Thm. 10]. *Suppose that $f(z)$ is a transcendental entire function, and set*

$$P = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log\log r}. \tag{1}$$

Given $\varepsilon > 0$,

$$B(r) > M(r)\left(1 - \frac{\pi^2(\sigma(P)+\varepsilon)}{2\log M(r)}\right), \quad -A(r) > M(r)\left(1 - \frac{\pi^2(\sigma(P)+\varepsilon)}{2\log M(r)}\right), \tag{2}$$

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