Commuting Toeplitz Operators on the Bergman Space of an Annulus

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Introduction

Let Ω be a domain in the complex plane $\mathbf{C}$, and let $L_a^2(\Omega)$ be the Bergman space consisting of those analytic functions on Ω that are square integrable on Ω with respect to area measure dA. Of particular interest are the cases $\Omega = D = \{z \in \mathbf{C} : |z| < 1\}$ and $\Omega = \mathfrak{A} = \{z \in \mathbf{C} : R < |z| < 1\}$ for $0 < R < 1$. The Bergman space is a closed subspace of the Hilbert space $L^2(\Omega)$ of all square integrable complex-valued functions on Ω, so there is an orthogonal projection P from $L^2(\Omega)$ onto $L_a^2(\Omega)$. If φ belongs to $L^\infty(\Omega)$, the Toeplitz operator with symbol φ, denoted T_φ, is a linear operator from $L_a^2(\Omega)$ to $L_a^2(\Omega)$ defined by $T_\varphi f = P(\varphi f)$. In [6] Axler and the author characterized commuting Toeplitz operators on $L_a^2(D)$ whose symbols are harmonic. A complex-valued function is harmonic on Ω if its Laplacian vanishes identically on Ω. We proved that two Toeplitz operators with symbols harmonic on D commute only in the obvious cases. In this paper we want to prove the analogous theorem for Toeplitz operators acting on $L_a^2(\mathfrak{A})$, provided their symbols are in a certain subclass of functions harmonic in $\mathfrak{A}$. It is well known that every function harmonic on D is of the form $f + \bar{g}$, where f and g are analytic on D. On the other hand, the logarithmic conjugation theorem [5, p. 179] implies that every u harmonic on $\mathfrak{A}$ is of the form $u(z) = f(z) + \bar{g}(z) + c\log|z|$, where f and g are analytic on $\mathfrak{A}$, $c \in \mathbf{C}$. Our commutativity theorem applies to harmonic symbols without the logarithmic terms. Namely, we have the following.

THEOREM 1. *Suppose that $\varphi = f_1 + \bar{f}_2$ and $\psi = g_1 + \bar{g}_2$ are bounded harmonic functions on $\mathfrak{A}$. Then $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if:*

(i) *φ and ψ are both analytic on $\mathfrak{A}$; or*
(ii) *$\bar{\varphi}$ and $\bar{\psi}$ are both analytic on $\mathfrak{A}$; or*
(iii) *there exist constants $a, b \in \mathbf{C}$, not both 0, such that $a\varphi + b\psi$ is constant on $\mathfrak{A}$.*

The main tool in the proof of the disk theorem was the automorphisms of the disk. However, the automorphisms of the annulus are very sparse and

Received October 5, 1995. Revision received December 23, 1995.
Michigan Math. J. 43 (1996).