On the Sum of Monotone Operators

LIANG-JU CHU

1. Introduction

Let X be a reflexive Banach space. Troyanski [28], using Asplund's averaging technique [1], has shown by means of a theorem of Lindenstrauss that there exists an equivalent norm on X that is everywhere Fréchet differentiable except at the origin and whose polar norm on its dual X^* is everywhere Fréchet differentiable except at the origin. For notational simplicity, we may assume throughout this paper that the given norm on X already has these special properties.

A set-valued operator $T: X \to X^*$ is a function sending each $x \in X$ to a (possibly empty) subset Tx of X^*. For a set-valued operator $T: X \to X^*$, the domain, the range, the graph, and the inverse of T are denoted, respectively, by

$$D(T) := \{x \in X; T(x) \neq \emptyset\},$$

$$R(T) := \bigcup\{x^* \in X^*; x^* \in T(x)\},$$

$$G(T) := \{(x, x^*) \in X \times X^*; x^* \in T(x)\},$$

and

$$T^{-1}(x^*) := \{x \in X; x^* \in T(x)\}.$$

Recall that a set-valued operator $T: X \to X^*$ is monotone, provided that

$$\langle x' - x, y' - y \rangle \geq 0 \quad \forall (x, y), (x', y') \in G(T).$$

T is maximal monotone provided T is monotone and there exists no other monotone set-valued operator whose graph properly contains the graph of T. Such operators have been studied extensively in both theory and applications; see, for example, the work by Brézis [3] and Phelps [19] and the references cited therein. It is known [20; 22; 23] that the subdifferential operator of a closed proper convex function is maximal monotone. The subdifferential operator of a convex function f on X is defined by

$$\partial f(x) := \{x^* \in X^*; f(z) - f(x) \geq \langle z - x, x^* \rangle \; \forall z \in X\}.$$

T is cyclically monotone, provided that

$$\langle x_1 - x_0, x_0^* \rangle + \langle x_2 - x_1, x_1^* \rangle + \cdots + \langle x_0 - x_m, x_m^* \rangle \leq 0$$

Received June 19, 1995. Revision received October 16, 1995.
Michigan Math. J. 43 (1996).