The Ahlfors Laplacian on a Riemannian Manifold with Boundary

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0. Introduction

In our previous paper [9] we initiated a study of the Ahlfors Laplacian $L = S^*S$, that is, the symmetric and trace-free part of the covariant derivative acting on vector fields on a Riemannian manifold M. Here S is the Ahlfors operator: $SX = 0$ means that the vector field X on M is conformal. The size of SX (in the infinity norm) measures the degree to which X is quasi-conformal (the constant of quasi-conformality). Thus a good understanding of the operators S and L is desirable in connection with studies of quasi-conformal transformations and their geometry.

In this paper we extend some of our earlier results to the case where M has a boundary $\Sigma = \partial M \neq \emptyset$. We investigate the behavior of S on a general hypersurface, thus relating the intrinsic conformal geometry of Σ with that of M. In this way we find geometrically natural boundary conditions for L, giving rise to self-adjoint and elliptic extensions of L up to the boundary. One such condition consists of the elasticity condition investigated by Weyl [16] in dealing with vibrations of an elastic body in the Eulidean space $\mathbb{R}^3$. We are thus able to generalize and sharpen the asymptotic distribution of eigenfrequencies found by Weyl, in a sense finding the "vibrational spectrum" of M. Note that L does not have scalar leading symbol, so that both the spectral asymptotics as well as the boundary conditions are a more delicate matter than, for example, for the Laplacian. Another question we address is that of unique continuation for conformal vector fields given a certain behavior on Σ; this is directly related to the existence of a Poisson kernel for L and for S.

It turns out that the basic formulas of Green's type for S and L are particularly simple. We derive these and show how similar formulas hold for any generalized gradient [14], based on a universal Green's formula for the covariant derivative. Such formulas were in a special case considered by Weyl and also by Yano in [17], where he used them to characterize conformal vector fields and their boundary values.

These general Green's formulas are remarkably simple and could in particular be applied to finding natural boundary conditions of a geometric

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