Extremal Problems for Quadratic Differentials

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1. Introduction

The Extremal Problem on the Unit Disk

Let $\mathbb{D}$ denote the unit disk and σ a finite set of four or more points on $\partial\mathbb{D}$. Then the Banach space Q_σ of all functions φ, holomorphic in $\mathbb{C}\backslash\sigma$, for which $\varphi(z)\,dz^2$ is real along $\partial\mathbb{D}\backslash\sigma$ and for which the L_1-norm

$$\|\varphi\| = \iint_{\mathbb{D}} |\varphi|\,dx\,dy < \infty,$$

has positive dimension. Let the homotopy types γ_j of cross-cuts in $\bar{\mathbb{D}}\backslash\sigma$ be given. By reflection across the boundary of $\mathbb{D}$ ($z \mapsto (\bar{z})^{-1}$), the cross-cuts γ_j become closed curves g_j on the Riemann sphere. We assume that the family of these closed curves on $\bar{\mathbb{C}}\backslash\sigma$ is *admissible* in the following sense:

(i) the curves g_j are nonintersecting Jordan curves;
(ii) no two of the closed curves g_j is homotopic in $\bar{\mathbb{C}}\backslash\sigma$; and
(iii) none of the curves g_j is homotopically trivial or homotopic to a single point of σ.

The system of cross-cuts γ_j is called admissible if the corresponding reflected system of closed Jordan curves g_j is admissible. The Banach space Q_σ is a real subspace of the complex vector space of holomorphic, quadratic differentials with finite norm on the Riemann surface $\bar{\mathbb{C}}\backslash\sigma$. Since there is a global parameter z for the Riemann surface $\mathbb{D}$, identifying $\varphi(z)$ with $\varphi(z)\,dz^2$ provides an isomorphism between functions and quadratic differentials. Using this identification, we will refer to elements of Q_σ as *quadratic differentials.*

Any quadratic differential on a Riemann surface S induces a vector of heights of homotopy classes of simple, closed curves on S. The height of a closed curve γ is the infimum of the integrals

$$\int_{\tilde{\gamma}} |\operatorname{Im}(\varphi(z)^{1/2}\,dz)|,$$

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