Functional Calculus for Noncommuting Operators

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1. Notation and Preliminaries

Throughout this paper, Λ stands for the set $\{1,2,\ldots,n\}$ where n is a fixed natural number. For every $k\in\mathbb{N}^*=\{1,2,\ldots\}$ let $F(k,\Lambda)$ be the set of all functions from the set $\{1,2,\ldots,k\}$ to Λ, and let

$$\mathfrak{F}=\bigcup_{k=0}^{\infty}F(k,\Lambda), \tag{1.1}$$

where $F(0,\Lambda)$ stands for the set $\{0\}$.

A sequence $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ of unilateral shifts on a Hilbert space $\mathcal{H}$ with orthogonal final spaces is called a Λ-*orthogonal shift* if the operator matrix $[S_1,S_2,\ldots,S_n]$ is nonunitary, that is, $\mathcal{L}:=\mathcal{H}\ominus(\bigoplus_{\lambda\in\Lambda}S_\lambda\mathcal{H})\neq\{0\}$. This definition is essentially the same as that from [4]. The dimension of $\mathcal{L}$ is called the *multiplicity* of the Λ-orthogonal shift. Two Λ-orthogonal shifts are *unitarily equivalent* if and only if they have the same multiplicity (see [6, Thm. 1.2]).

Let us consider a model Λ-orthogonal shift of multiplicity 1, acting on the full Fock space [3]

$$F^2(H_n)=\mathbf{C}1\oplus\bigoplus_{m\geq1}H_n^{\otimes m}, \tag{1.2}$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1,e_2,\ldots,e_n\}$.

For each $\lambda\in\Lambda$ we define the isometry S_λ by

$$S_\lambda h=e_\lambda\otimes h\quad\text{for }\ h\in F(H_n). \tag{1.3}$$

It is easy to see that $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ is a Λ-orthogonal shift of multiplicity 1. This model will play an important role in our investigation. We shall denote by $\mathcal{P}$ the set of all $p\in F^2(H_n)$ of the form

$$p=a_0+\sum_{\substack{1\leq i_1,\ldots,i_k\leq n\\1\leq k\leq m}}a_{i_1\ldots i_k}e_{i_1}\otimes\cdots\otimes e_{i_k},\quad m\in\mathbf{N},$$

where $a_0,a_{i_1\ldots i_k}\in\mathbf{C}$. The set $\mathcal{P}$ may be viewed as the algebra of the polynomials in n noncommuting indeterminates, with $p\otimes q$, $p,q\in\mathcal{P}$, as multi-

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