Efficient Representatives for Automorphisms of Free Products

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The Scott conjecture for automorphisms of free groups says that if ϕ is an automorphism of a free group of rank n, then the subgroup of elements fixed by ϕ is a free group of rank at most n. This conjecture was recently settled in the positive in the brilliant paper of Bestvina and Handel [BH], which applied the Perron–Frobenius theory of nonnegative matrices [Se], the folding techniques introduced by Stallings [St], and motivations from the train-track theory of homeomorphisms of surfaces to the study of self-homotopy equivalences of graphs. In this paper we prove the generalization of the Scott conjecture to an arbitrary group G represented as a free product of freely indecomposible factors. Our proof is patterned on that of Bestvina and Handel: we consider 2-complexes $\mathfrak{X}$ whose fundamental groups are isomorphic to G and *efficient* self-homotopy equivalences $f: \mathfrak{X} \to \mathfrak{X}$ (see 2.10) that generalize the relative train-track maps of [BH]. In Section 1, we define topological maps of graphs of complexes, describe the way in which they model automorphisms of free products, and discuss the simplification operations on them. In Section 2 we define and prove the existence of efficient representatives for general automorphisms (see 2.12) and establish their most important properties. In Section 3, we apply the analysis of Section 2 to construct a "core" $\mathfrak{F}$ of the covering space $\hat{\mathfrak{X}}$ of $\mathfrak{X}$ corresponding to the subgroup $\mathrm{Fix}(\phi)$ (see 3.8) and prove that its fundamental group has Kuroš rank at most that of the group (3.11). This was the approach employed in [GT] (resp. [CT]) in proving the finite rank of $\mathrm{Fix}(\phi)$ in the free (resp. free product) case.

1. Basic Objects and Constructions

In order to generalize the Scott conjecture, we need general notions of the (absolute) rank of a group and the (relative) rank of a subgroup (groups will always be assumed to be countable). If $G = \bigstar_{i=1}^{m} G_i$ is represented as a free product of freely indecomposable factors, then it was shown in [Ku] that the set of factors (and in particular the number of factors) is well-defined up to isomorphism. The Kuroš subgroup theorem states that if H is a subgroup

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