n-Harmonic Morphisms in Space Are Möbius Transformations

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1. Introduction

Let Ω and Ω' be open subsets of $\mathbf{R}^n$. A mapping $f: \Omega \to \Omega'$ is a harmonic *motion* if whenever $u: \Omega' \to \mathbf{R}^n$ is a harmonic function so too is $v = u \circ f$. We say that f is a harmonic *morphism* if f is continuous and $u \circ f$ is harmonic in $f^{-1}(\mathbf{D})$ whenever $u: \mathbf{D} \to \mathbf{R}$ is harmonic, where $\mathbf{D}$ is an arbitrary open subset of Ω'. Clearly every harmonic morphism is a harmonic motion.

It is easy to see that in the one dimension $n = 1$, harmonic motions are simply affine functions. When $n \geq 2$, by considering the harmonic functions x_i, $i = 1, \ldots, n$, and $x_i x_j$, $x_i^2 - x_j^2$ for $i \neq j$, we obtain that each component f_i of f is harmonic, $\langle \nabla f_i, \nabla f_j \rangle = 0$, and $|\nabla f_i| = |\nabla f_j|$ for $i \neq j$. Therefore the differential of f, $Df(x)$, must be a conformal matrix for all $x \in \Omega$. In two dimensions $n = 2$ we conclude that f is analytic or anti-analytic on each component of Ω.

In higher dimensions $n \geq 3$, a classical theorem of Liouville implies that f must be a Möbius map on each component of Ω. Therefore, f can be expressed as a finite composition of similarities (rotations, translations, and reflections on planes) and inversions on spheres. Since these last inversions are not harmonic we conclude that, on each component of Ω,

$$f(x) = \lambda \Theta x + b \tag{1}$$

for certain $\lambda \in \mathbf{R}$, $b \in \mathbf{R}^n$, and Θ an orthogonal matrix. In particular, note that in this Euclidean case harmonic motions are also harmonic morphisms.

More details can be found in [GH], where homeomorphic motions are treated and nonpositive metrics are included, and in [Fu] and [Is], where harmonic morphisms between Riemannian manifolds are studied. Motions of linear partial differential equations with constant coefficients are studied in [Ru]. Finally, a treatment of harmonic morphism from the point of view of abstract potential theory is in [CC].

In this paper we study p-harmonic morphisms. These are defined as above by requiring that they preserve p-harmonic functions, which are solutions of the p-Laplace equation

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