Equivariant Simple Poincaré Duality

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Introduction

In this paper we describe, for a finite transformation group G, the simple homotopy theory of cell complexes whose cells are discs of representations. Our primary goal is to define equivariant simple Poincaré complexes and to prove a π–π theorem for equivariant simple Poincaré pairs.

There is in the literature on equivariant surgery much about the question: When is a degree-1 normal map between G-manifolds normally cobordant to a G-equivalence or to a simple G-equivalence? (See [DP], [DR] and [LM]; for a comprehensive overview see [DS].) In this paper we begin to address the question: When is a G–CW complex simply G-homotopy equivalent to a smooth G-manifold? If we drop the requirement that the equivalence be simple, some results are given in [CW1] and [CW2].

The first obstruction to discussing the case of simple equivalence is the question of what is meant by a simple G-Poincaré complex, and behind this is the question of what is meant by G-Poincaré duality. It has been customary [DR; Lü] to consider a nonequivariant triangulation in which G permutes the simplices; the nonequivariant chains then have an induced $\mathbf{Z}G$-action and nonequivariant Poincaré duality gives an equivalence of $\mathbf{Z}G$ chains. This equivalence fails in general to be a simple $\mathbf{Z}G$ equivalence, and in fact Lück defines the Poincaré torsion of a smooth G-manifold which measures this failure [Lü, 18.G]. Thus, under this interpretation, smooth G-manifolds are usually not simple Poincaré complexes, and so a theory of simple G-Poincaré complexes would seem pointless.

From the point of view of equivariant stable homotopy theory there is another problem. From this point of view "ordinary homology" means Bredon's ordinary equivariant homology theory [Br]. This uses the same CW decomposition as above, but the Bredon chains incorporate more information about the fixed sets of the G-complex in question. In this theory we do not even have Poincaré duality itself for smooth G-manifolds, except in very special cases such as free actions or trivial actions.

We can resolve both these difficulties if we redefine what we mean by equivariant Poincaré duality. It is well known that it is useful to extend the

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