Covolume Estimates for Discrete Groups of Hyperbolic Isometries Having Parabolic Elements

SA'AR HERSONSKY

1. Introduction

In this paper we address the problem of estimating the covolume of a discrete group of orientation-preserving isometries of hyperbolic space in any dimension. We provide a general method for obtaining a lower bound for the covolume in the case that the group has parabolic elements. For a torsion-free discrete group of orientation-preserving isometries of hyperbolic 4-space having parabolic elements, we give an explicit lower bound.

By the Bieberbach theorem (see [13, Thms. 3.2.1 and 3.2.2]), for any given dimension n there are only finitely many cocompact discrete groups of isometries of $\mathbf{R}^n$, up to affine equivalence. These are called the n-dimensional Bieberbach groups. Each n-dimensional Bieberbach group G contains a finite-index subgroup $A = A_G$ isomorphic to Z^n. The subgroup A_G consists of all translations in G. Let us denote by I_n the maximum of the indices $|G:A_G|$, where G ranges over all n-dimensional orientation-preserving Bieberbach groups.

If Λ is a lattice in $\mathbf{R}^n$, we denote by $|\Lambda|$ the Euclidean volume of $\mathbf{R}^n/\Lambda$ and by β the nonzero vector of smallest length in Λ. It is well known that for every $n>0$ there exists a positive constant δ_n such that for any lattice Λ in $\mathbf{R}^n$ we have $|\Lambda| \geq \delta_n |\beta|^n$. (See [3] for the values of δ_n when $n \leq 8$.)

Our first main result is the following theorem.

THEOREM 1. *Let Δ be a discrete subgroup of orientation-preserving isometries of hyperbolic $(n+1)$-space $\mathbf{H}^{n+1}$. Suppose that Δ has finite covolume. Let m denote the number of orbits of points in $\hat{\mathbf{R}}^n$ which are fixed by parabolic elements of Δ. Then $m<\infty$ and*

$$\operatorname{vol}\left(\frac{\mathbf{H}^{n+1}}{\Delta}\right) \geq \frac{\delta_n m}{n I_n}.$$

Section 3 of this paper is devoted to the proof of this theorem.

Our second main result deals with the 4-dimensional torsion-free case, and will be stated in terms of hyperbolic manifolds. If Δ is a discrete, torsion-free

Received January 14, 1992. Revision received February 15, 1993.
Michigan Math. J. 40 (1993).