Removable Singularities for Analytic Functions

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1. Introduction

The question of removable singularities for analytic functions which are of bounded mean oscillation (in BMO) or uniformly Hölder continuous with exponent α, $0<\alpha<1$, (in Lip_α) is well understood (see e.g., [Gn, 4.5; Kr; Ka]). In these cases a more or less complete answer can be given in terms of the Hausdorff dimension: a compact subset E of a domain G in the complex plane $\mathbf{C}$ is removable for analytic functions defined in $G \setminus E$ and belonging to $\mathrm{BMO}(G)$ ($\mathrm{Lip}_\alpha(G)$) if and only if $\mathrm{H}^1(E)=0$ ($\mathrm{H}^{1+\alpha}(E)=0$). Here H^β denotes β-dimensional Hausdorff measure.

In this note we consider the analogous question for analytic functions defined in $G\setminus E$ and belonging to $\mathrm{BMO}(G\setminus E)$ or $\mathrm{locLip}_\alpha(G\setminus E)$—that is, instead of assuming a regularity condition in all of G, we require only that our analytic functions satisfy a regularity condition on $G\setminus E$. Recall that if U is an open set in $\mathbf{C}$, then a complex-valued function f belongs to $\mathrm{BMO}(U)$ if there is a constant M such that

$$|B|^{-1}\int_B |f(z)-f_B|\,dx\,dy \le M$$

for each open disc $B\subset U$, where $f_B=|B|^{-1}\int_B f(z)\,dx\,dy$ and $|B|$ is the area of B. Next, suppose that $0<\alpha\le 1$. Following [GM], we say that f belongs to $\mathrm{locLip}_\alpha(U)$ if there is a constant M such that

$$|f(z)-f(w)|\le M|z-w|^\alpha$$

whenever z, w belong to a disc B contained in U. Finally, we recall the definition of the Minkowski dimension of a compact set $K\subset \mathbf{C}$. For $\lambda>0$ and $r>0$ write

$$M_r^\lambda(K)=\inf\left\{kr^\lambda : K\subset \bigcup_{i=1}^k B(z_i;r)\right\}$$

and let

$$M^\lambda(K)=\limsup_{r\to 0} M_r^\lambda(K).$$

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