The Inner Carathéodory Distance for the Annulus II

MAREK JARNICKI & PETER PFLUG

Let

$$A = \{\lambda \in \mathbf{C} : 1/R < |\lambda| < R\} \quad (R > 1),$$

and let c_A, c_A^i denote the *Carathéodory distance* and the *inner Carathéodory distance* for the annulus A, respectively (cf. [3]). It is known that $c_A \neq c_A^i$ (cf. [2; 4])—more precisely, for any $\lambda', \lambda'' \in A$, the following equivalence is true:

$$c_A(\lambda', \lambda'') = c_A^i(\lambda', \lambda'') \quad \text{if and only if } \lambda' \text{ and } \lambda'' \text{ lie on the same radius, i.e., } \arg \lambda' = \arg \lambda'' \quad (\text{cf. [4]}). \tag{1}$$

Recall (cf. [1]) that

$$c_A^i(\lambda', \lambda'') = \inf\{L_{\gamma_A}(\alpha) : \alpha : [0,1] \to A \text{ is a piecewise } C^1\text{-curve with } \alpha(0) = \lambda',\ \alpha(1) = \lambda''\}, \tag{2}$$

where $L_{\gamma_A}(\alpha)$ denotes the γ_A-length of α given by the formula

$$L_{\gamma_A}(\alpha) = \int_0^1 \gamma_A(\alpha(\vartheta); \alpha'(\vartheta))\, d\vartheta. \tag{3}$$

In (3), $\gamma_A : A \times \mathbf{C} \to \mathbf{R}_+$ denotes the *Carathéodory-Reiffen metric* for A.

It is known (cf. [6]) that

$$\gamma_A(\lambda; X) = \frac{1}{R|\lambda|^2} \cdot f\left(\frac{1}{|\lambda|}, -|\lambda|\right) \cdot \Pi(|\lambda|, |\lambda|) \cdot |X| \tag{4}$$

for λ in A and X in $\mathbf{C}$, where

$$f(s, \lambda) = \left(1 - \frac{\lambda}{s}\right) \cdot \Pi(s, \lambda) \tag{5}$$

and

$$\Pi(s, \lambda) = \frac{\prod_{n=1}^{\infty}(1 - (\lambda/s)R^{-4n})(1 - (s/\lambda)R^{-4n})}{\prod_{n=1}^{\infty}(1 - \lambda s R^{-4n+2})(1 - (1/\lambda s)R^{-4n+2})} \tag{6}$$

for $1/R < s < R$ and $\lambda \in A$.

The aim of this note is to provide effective formulas for c_A^i—more precisely, for any $\lambda', \lambda'' \in A$, we will find an effective description of the *shortest*

Received October 8, 1991. Revision received November 8, 1992.
Michigan Math. J. 40 (1993).