Conditioned Brownian Motion and Hyperbolic Geodesics in Simply Connected Domains

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0. Introduction

Let D be a simply connected planar domain. Let B_t be Brownian motion in D with lifetime τ_D. If h is a positive harmonic function in D, then the Brownian motion conditioned by h is determined by the transition functions

$$P_t^h(w,z) = \frac{1}{h(w)} P_t^D(w,z)\, h(z),$$

where $P_t^D(w,z)$ are the transition functions for the unconditioned Brownian motion in D. We let P_w^h be the measure on path space induced by the P_t^h and write E_w^h for the corresponding expectation. If $h=1$, the case of killed Brownian motion in D, we simply write P_w and E_w. The following result is due to Cranston and McConnell [9].

THEOREM A. *Let D be any planar domain and denote by $H^+(D)$ the collection of all positive harmonic functions in D. Then*

$$\sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D) \le C\,\mathrm{area}(D), \tag{0.1}$$

where C is a universal constant.

This result has been extended in several directions. We refer the reader to Bañuelos [6], where a survey of the recent literature on this subject is given. The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let D be a simply connected planar domain. If $z=x+iy\in D$ and if Γ is a geodesic for the hyperbolic metric in D, we let $d_D(z,\Gamma)$ be the hyperbolic distance from z to Γ. There are universal constants c_1 and c_2 such that*

$$c_1 \sup_{\Gamma} \iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy \le \sup_{\substack{w\in D\\ h\in H^+(D)}} E_w^h(\tau_D) \le c_2 \sup_{\Gamma} \iint_D e^{-2d_D(z,\Gamma)}\,dx\,dy,$$

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