Examples of Nonproper Affine Actions

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0. Introduction

Margulis ([Ma1], [Ma2]) was the first to show that complete affinely flat manifolds with free fundamental groups of rank ≥ 2 exist. These "Margulis space-times" correspond to free subgroups of the affine group that act properly discontinuously on $\mathbf{R}^3$. These subgroups of the affine group are conjugate to free subgroups of $\mathcal{V} \rtimes \mathbf{G} = \mathbf{H}$, where $\mathcal{V}$ is the group of parallel translations in $\mathbf{R}^3$ and $\mathbf{G} = SO^0(2,1)$ [FG].

Margulis also presented in [Ma1] and [Ma2] a test that identifies some free subgroups of $\mathbf{H}$ that do not act properly discontinuously on $\mathbf{R}^3$. Although Margulis's constraints are necessary to ensure that a free subgroup of $\mathbf{H}$ acts properly discontinuously on $\mathbf{R}^3$, it will be shown here that they are not sufficient. Margulis's proof that these constraints are necessary will also be presented here, to correct the many mistakes that are in the translation and to isolate and clarify the ideas in the proof as they appear in [Ma1] and [Ma2].

1. Geometry of $\mathbf{R}^{2,1}$

Consider subgroups of the affine group of the form $\Gamma = \langle h_1, h_2 \rangle \subset \mathbf{H}$. Γ acts on $\mathcal{E} = \mathbf{R}^{2,1}$ with its inner product $\mathbf{B}(x,y) = x_1 y_1 + x_2 y_2 - x_3 y_3$ invariant under the action of $\mathbf{G}$. For the null cone $C = \{x \in \mathcal{E} \mid \mathbf{B}(x,x) = 0\}$, let $W = \{x \in C \mid x_3 > 0\}$. Also, the Euclidean length of a vector is

$$\|x\| = ({x_1}^2 + x_2^2 + x_3^2)^{1/2}$$

and the Euclidean distance between two vectors is $\rho(x,y) = \|x-y\|$ or (more generally) the Euclidean distance between two sets

$$\rho(A,B) = \min\{\rho(a,b) \mid a \in A \text{ and } b \in B\}.$$

Note that if z is the vector $\{x_1, x_2, -x_3)$ then $\mathbf{B}(x,y)$ is equal to the Euclidean inner product of z and y and the Lorentzian Schwartz inequality, $\mathbf{B}(x,y) \leq \|z\|\|y\| = \|x\|\|y\|$, holds.

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