# On Similarity of Operators to Isometries

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## 1. Introduction

The problem of the similarity of the operators on Hilbert spaces to isometric operators has been considered by Sz.-Nagy [4]. He proved that an operator $A$ on a Hilbert space $\mathcal{3C}$ is similar to an isometric operator if and only if there exist $a$ and $b > 0$ such that $b\|h\| \leq \|A^n h\| \leq a\|h\|$ for each $h \in \mathcal{3C}$, $n \in \mathbf{N}$.

In particular, an operator $A$ is similar to a unitary operator if and only if $A$ is invertible and

$$\sup\{\|A^n\|; n \in \mathbf{Z}\} < \infty.$$

Other necessary and sufficient conditions for the similarity to a unitary operator have been obtained [5; 7; 2; 9]. Let us recall from [9] one such condition that will be used in this paper: A power bounded operator $A$ on $\mathcal{3C}$ is similar to a unitary operator if and only if $A$ is surjective and if there exists $c > 0$ such that $\|(A - \lambda)h\| \geq c(1 - |\lambda|)\|h\|$, for each $\lambda \in \mathbf{D} = \{z \in C : |z| < 1\}$ and each $h \in \mathcal{3C}$.

Concerning the similarity of a contraction $T$ to an isometric operator, necessary and sufficient conditions have been obtained [6; 10] in terms of the characteristic function of $T$. Recently, Uchiyama [8] has also obtained new criteria for contractions to be similar to isometries.

In [1], Fadeev gives some (necessary *or* sufficient) conditions for the similarity of the contractions to isometric operators in terms of their resolvents. Also, an example illustrating the precision of his conditions is given. The aim of this paper is to give some necessary *and* sufficient conditions under which an operator $A$ on a Hilbert space $\mathcal{3C}$ is similar to an isometric operator or to a unilateral shift, in terms of the resolvent of the operator.

In Sections 3, 4, and 5, we shall provide some new necessary and sufficient conditions for a contraction $T$ to be similar to an isometry. It is also shown that if $T$ is similar to an isometry then it is similar to a restriction of the minimal unitary dilation of $T$ to a certain invariant subspace.

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