On the Valence Structure of Analytic Functions

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Let f be an analytic function in the open unit disk $\mathbf{D}$. The *valence function* is defined by

$$\nu_f(w) = \operatorname{card}[f^{-1}(w) \cap \mathbf{D}], \quad w \in \mathbf{C},$$

where pre-images are counted with multiplicities. In this article we give a negative answer to the following question posed by Stephenson [5, Question 2]. We say that a function is *analytic on the closed unit disc $cl\mathbf{D}$* if it is analytic in some neighbourhood of $cl\mathbf{D}$.

QUESTION. If f and g are analytic on $cl\mathbf{D}$ with identical valence functions, does there exist an algebraic homeomorphism ψ of $\partial\mathbf{D}$ with $f \circ \psi \equiv g$?

THEOREM 1. *There exist f and g analytic on $cl\mathbf{D}$ such that $\nu_f(w) \equiv \nu_g(w)$, $w \in \mathbf{C}$, but $f \circ \psi \not\equiv g$ for any homeomorphism ψ of the unit circle $\partial\mathbf{D}$.*

The proof is based on the following theorem.

THEOREM 2. *There exists a function ϕ analytic on $cl\mathbf{D}$ and two disjoint arcs $I_1, I_2 \subset \partial\mathbf{D}$ such that ϕ maps each of them homeomorphically onto the same arc but with opposite orientations.*

Note that if ϕ is not required to be analytic across $\partial\mathbf{D}$ then such a function can be constructed easily. (Take, for example, $\phi(z) = \omega^2(z)$, where $\omega(z)$ is a conformal map of $\mathbf{D}$ onto $\{z \in \mathbf{D} : \operatorname{Re} z > 0\}$.) To some extent this is also true for Theorem 1: Stephenson [5] produced two analytic functions in $\mathbf{D}$, piecewise analytic and continuous on $\partial\mathbf{D}$, with all other properties of f and g in Theorem 1.

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Before we proceed to the proof, let us indicate some relations between Theorem 2 and the multiplicity of analytic Toeplitz operators. Recall that for $\phi \in H^\infty$ the Toeplitz operator T_ϕ is the multiplication operator on the Hardy

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