Hankel and Toeplitz Operators on the Fock Space

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1. Introduction

Throughout this paper let $n \in \mathbf{N}$ be fixed. Let μ be the Gaussian measure on $\mathbf{C}^n$ defined by $d\mu(z) = e^{-\|z\|^2/2} dV(z)/(2\pi)^n$, where V is the usual Lebesgue measure on $\mathbf{C}^n$. The *Fock space* $\mathfrak{F}$, also called the Segal–Bargmann space, is the set of holomorphic functions which are in $L^2(\mathbf{C}^n, \mu)$. The Fock space $\mathfrak{F}$ is a closed subspace of the Hilbert space $L^2(\mathbf{C}^n, \mu)$, with inner product given by $\langle f, g\rangle = \int_{\mathbf{C}^n} f(z)\overline{g(z)}\, d\mu(z)$ for $f, g \in L^2(\mathbf{C}^n, \mu)$. Let P denote the orthogonal projection of $L^2(\mathbf{C}^n, \mu)$ onto $\mathfrak{F}$. For a function $f \in L^\infty(\mathbf{C}^n)$, the *Toeplitz operator* $T_f \colon \mathfrak{F} \to \mathfrak{F}$ and the *Hankel operator* $H_f \colon \mathfrak{F} \to \mathfrak{F}^\perp$ are defined by

$$T_f g = P(fg), \quad g \in \mathfrak{F},$$

$$H_f g = (I-P)(fg), \quad g \in \mathfrak{F}.$$

It is clear that these are bounded operators for every function $f \in L^\infty(\mathbf{C}^n)$. Berger and Coburn [1] characterized the functions $f \in L^\infty(\mathbf{C}^n)$ for which H_f is compact, and also obtained the result that H_f is compact if and only if $H_{\bar f}$ is. In this paper we will give an alternate approach to Berger and Coburn's work. Our method is more elementary and furthermore it also gives the functions $f \in L^\infty(\mathbf{C}^n)$ for which T_f is compact. Writing τ_λ to denote the translation on $\mathbf{C}^n$ by λ, we will prove that the Hankel operator H_f is compact if and only if $\|f \circ \tau_\lambda - P(f \circ \tau_\lambda)\|_2 \to 0$ as $\|\lambda\| \to \infty$. This result is completely analogous to the author's characterization of the compact Hankel operators on the Bergman spaces of the unit disk [6], and the unit ball and polydisk in $\mathbf{C}^n$ [7]. We will show how this result implies Berger and Coburn's result.

The paper is arranged as follows. In Section 2 we give the preliminaries needed for the rest of the paper. In Section 3 we give the proof of our main result, characterizations of compact Hankel and Toeplitz operators. In Section 4 we obtain Berger and Coburn's result that H_f is compact if and only if $H_{\bar f}$ is. In Section 5 we consider Hankel operators with bounded continuous symbols. For a subclass of these Hankel operators we formulate a very useful criterium for compactness. As an immediate consequence we obtain another proof of Berger and Coburn's result mentioned above. In Section 6

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