Removable Sets for Harmonic Functions

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0. Introduction

Suppose that K is a compact subset of $\mathbf{R}^d$, $d \geq 2$. A theorem of Carleson [Ca, Thm. VII.2] states that K is removable for harmonic functions satisfying a Lip_α condition, $0 < \alpha < 1$, if and only if $m_{d-2+\alpha}(K) = 0$; here m_β denotes β-dimensional Hausdorff measure.

Carleson's result fails for $\alpha = 1$: While it is easy to see from Green's theorem that K is removable for Lip_1 harmonic functions if $m_{d-1}(K) = 0$, Uy [Uy] has recently given an example of a compact subset of $\mathbf{R}^d$ that is removable for Lip_1 harmonic functions in spite of having positive $(d-1)$-dimensional measure. (As noted in [Uy], for $d = 2$ this follows from the existence of a set of positive length that is removable for bounded holomorphic functions. Such an example was given by Vitushkin [Vt] and simplified by Garnett [Gt]; the example in [Uy] is a generalization to $\mathbf{R}^d$ of the example in [Gt].)

We shall show that K is removable for harmonic functions in the Zygmund class if and only if $m_{d-1}(K) = 0$. (Definitions and a more precise statement follow.) The argument below may also be used to give a proof of Carleson's theorem for $0 < \alpha < 1$ which is perhaps somewhat simpler than the argument in [Ca].

Suppose Ω is an open subset of $\mathbf{R}^d$ and $u : \Omega \to \mathbf{R}$. We say that u is a *Zygmund function* on Ω ($u \in \Lambda_1(\Omega)$) if u is continuous on Ω and there exists $c < \infty$ such that

$$|u(x-y) - 2u(x) + u(x+y)| \leq c|y| \tag{0}$$

whenever $x, x \pm y \in \Omega$.

Note that the hypothesis of continuity cannot be omitted here; (0) alone does not imply that u is measurable, even for $\Omega = \mathbf{R}^d$ [Kr]. However, it is easy to see that u must be continuous if it is upper semicontinuous and satisfies (0), so that in particular a subharmonic function satisfying (0) is a Zygmund function. (We should perhaps point out that the standard definition of Λ_1 requires that u be (globally) bounded [Kr]; we shall find it more convenient to omit this condition.)

Received September 7, 1990. Revision received December 3, 1990.
Michigan Math. J. 38 (1991).