Compact Contractive Projections in Continuous Function Spaces

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The problem of characterizing subspaces of $\mathcal{C}(K)$ admitting contractive projections has been considered by different authors. From Nachbin, Goodner, and Kelley's theorems we obtain that if K is an extremely disconnected compact Hausdorff space, then a closed subspace of $\mathcal{C}(K)$ is the range of a projection of norm 1 in $\mathcal{C}(K)$ if and only if it is isometric to the continuous functions on an extremely disconnected compact Hausdorff space. Lindenstrauss and Wulbert [4] extended this result when K is a compact Hausdorff space, showing that a Banach space Y is isometric to the range of a contractive projection in some $\mathcal{C}(K)$ if and only if Y is isometric to $\mathcal{C}_\sigma(L) := \{f \in \mathcal{C}(L) : f(x)+f(\sigma(x)) = 0\ \forall x \in L\}$ for some compact Hausdorff space L and some involutive homeomorphism σ of L. Later on, Lindberg [2] gave necessary and sufficient conditions for a closed separating subspace E of $\mathcal{C}(K)$ to be the range of a projection of norm 1, obtaining that each contractive projection onto E can be given in terms of a real-valued continuous function defined on the closure of the single extreme points and the closure of the double extreme points.

In this paper we attempt to discuss the conditions for a separating subspace of $\mathcal{C}(X)$ to be the range of a compact contractive projection in $\mathcal{C}(X)$, X being a Hausdorff completely regular topological space with a fundamental sequence of compact sets.

Throughout this paper X will stand for any Hausdorff completely regular topological space and $\mathcal{C}(X)$ for the space of the continuous real-valued functions on X endowed with the compact-open topology. Given a linear subspace E of $\mathcal{C}(X)$ and a compact subset K of X, we shall set $E_K := \{f \in E : |f(x)| \le 1\ \forall x \in K\}$ and $C_K := \{f \in \mathcal{C}(X) : |f(x)| \le 1\ \forall x \in K\}$, and denote by E_K° and C_K° their polar sets in the topological dual spaces of E and $\mathcal{C}(X)$, respectively. E is said to be *separating* if, for each $x, y \in X$, $x \ne y$, there is some $f \in E$ such that $f(x) \ne f(y)$. E separates points and closed sets of X if, for each closed subset A of X and $x \in X \setminus A$, there is some $f \in E$ such that $f(x) \notin \overline{f(A)}$. For each $x \in X$, δ_x will denote the linear form of $(\mathcal{C}(X))'(E')$ such that $\delta_x(f) = f(x)\ \forall f \in \mathcal{C}(X)(E)$. If A is a subset of

Received February 2, 1990. Revision received November 28, 1990.
Michigan Math. J. 38 (1991).