A Formula for the Local Dirichlet Integral

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Dedicated to the memory of Allen L. Shields

1. Introduction

Let $H^2(\mathbf{D})$ denote the Hardy space of the open unit disc $\mathbf{D}$. The isometric isomorphism of $H^2(\mathbf{D})$ onto the closed subspace H^2 of L^2 of the unit circle is a map that is well understood. In fact, much of our knowledge about H^2 functions has been derived by an exploitation of the properties of this map.

The Dirichlet space D is the space of analytic functions f in $\mathbf{D}$ with finite Dirichlet integral; that is,

$$D(f)=\iint_{\mathbf{D}}|f'(z)|^2\,dA(z)<\infty,$$

where $dA(re^{it})=(1/\pi)r\,dr\,dt$ denotes the normalized area measure on $\mathbf{D}$. It is well known (and easy to verify) that D is contained in $H^2(\mathbf{D})$. It thus follows that the above-mentioned isomorphism maps D into a subset of L^2, and for problems involving variations within the class of Dirichlet functions it is important to know how changes in the boundary values affect the Dirichlet integral.

In his investigations about minimal surfaces, Douglas [9] used the following formula for the Dirichlet integral of f:

$$D(f)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(e^{it})-f(e^{is})}{e^{it}-e^{is}}\right|^2 dt\,ds.$$

This or similar formulas have also been used by other authors; for example, Beurling used one in his original proof [3] that the Fourier series of a Dirichlet function converges everywhere except perhaps on a set of logarithmic capacity zero. In 1960 Carleson proved a formula that expresses the Dirichlet integral of f as a sum of three nonnegative terms, involving respectively the Blaschke factor of f, the singular inner factor, and the outer factor (see [6] and also Corollary 3.6 below). As one application of this we mention a result of Brown and Shields, who used Carleson's formula to show that the "cut-off" operation, which maps a Dirichlet space function f to the outer function defined by $|g|=\min\{|f|,1\}$ on the unit circle $\mathbf{T}$, does not increase the Dirichlet integral (see [4, p. 284]).

Received December 11, 1989. Revision received November 21, 1990.
*Research supported in part by the National Science Foundation.
Michigan Math. J. 38 (1991).