A Functional Calculus for a Scalar Perturbation of $\partial/\partial z$

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1. Introduction

In this paper, we determine when a functional calculus exists for the operator

$$L = a_1\left(-i\frac{\partial}{\partial z}\right) + a_2\left(-i\frac{\partial}{\partial \bar{z}}\right), \quad a_1 \text{ close to } 1,\ a_2 \text{ close to } 0.$$

In other words, we consider when $\phi(L)$ can be defined as a bounded operator on $L^2(\mathbf{R}^2)$ for a certain class of functions ϕ. The operator L is not normal, thus the usual spectral theory cannot be applied. The spectrum of L is the whole complex plane, so resolvents need to be interpreted, and one cannot define functions of L by integrating on the boundary of the spectrum.

Extending the unpublished results of Coifman and Meyer ([CM2]; see also [CM1]), we construct a functional calculus for L and prove L^2 boundedness for a certain class of ϕ, and connect the study of the functional calculus to a certain surface in $\mathbf{C}^2$. The assumption of the boundedness on L^2 of some natural functions of L is equivalent to certain quantitative conditions on the surface. We also show how L can be obtained by conjugation from the Coifman–Meyer case. This gives another geometric interpretation: a connection via a change of variables to a simpler surface considered by Coifman and Meyer.

In Section 2, we discuss some general facts about functional calculi which lead to the definition of a surface Σ in $\mathbf{C}^2$ and the definition of the conjugate operator $\bar{L}$. Section 3 examines restrictions on the coefficients a_1 and a_2, and exhibits a class of functions satisfying these restrictions. In Section 4, we calculate $\bar{L}/L$ and $L/\bar{L}$, while in Section 5 we use the expression

$$\phi(L) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial\phi}{\partial\bar{\xi}}\frac{1}{L-\xi}\,d\sigma(\xi)$$

to define $\phi(L)$ for $\varphi \in C_0^\infty(\mathbf{C})$. In Section 6 we show that the product formula holds for the functional calculus, and in Section 7 we extend the class of ϕ

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