Landau and Schottky Theorems for Holomorphic Curves

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The theorems of Landau and Schottky on holomorphic functions were generalized to holomorphic curves in the projective plane in a paper published in 1926 by Bloch [1]. A gap in Bloch's argument was filled by Cartan in his thesis published in 1928 [4]. The aim of this paper is to prove Bloch's versions of the Landau and Schottky theorems with explicit expressions for the constants. It appears that this has never been done before. In contrast, very precise expressions for the constants in the classical Landau and Schottky theorems have been obtained by a succession of authors, of whom the most recent is Hempel [9].

We shall write $B(a, r)$ for the open ball of centre a and radius r, and $\bar{B}(a,r)$ for the closed ball of centre a and radius r. The open annulus of centre a, inner radius r and outer radius R will be denoted by $A(a,r,R)$.

The classical Schottky theorem concerns a holomorphic function f: $B(0,1)\to\mathbf{C}$. Suppose that f omits the values 0 and 1 as well as the value ∞. Then, for any ρ satisfying $0<\rho<1$, there is an upper bound for $|f(z)|$ when $z\in\bar{B}(0,\rho)$, depending only on ρ and $f(0)$. It follows easily that under the same hypotheses there is an upper bound for $|f'(0)|$ depending only on $f(0)$, and this is the content of Landau's theorem.

To describe Bloch's version of the Landau and Schottky theorems, it is convenient to work in homogeneous coordinates on $\mathbf{CP}^2$. Adopting Bloch's notation, we let X, Y, Z, T be four holomorphic functions on $B(0,1)$. We write $X_0=X(0)$, $Y_0=Y(0)$, $Z_0=Z(0)$, and $T_0=T(0)$. We shall usually suppress the notation for the point at which one of these functions is evaluated. We assume that X, Y, Z, T have no zeros and satisfy the identity

$$X+Y+Z+T=0.$$

We may regard (X,Y,Z) as an expression in homogeneous coordinates (x^0, x^1, x^2) for a holomorphic curve $c: B(0,1)\to\mathbf{CP}^2$. The curve c omits four lines in general position with the equations

$$x^0=0,\quad x^1=0,\quad x^2=0,\quad x^0+x^1+x^2=0.$$

We seek bounds for the ratios of X, Y, Z, and T, or the derivatives of these ratios, in terms of X_0, Y_0, Z_0, and T_0. It is obvious that there are some cases

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