Boundary Density and the Green Function

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In this note, we generalize the following theorem on level curves of conformal mappings to domains in $\mathbf{R}^m$, $m \geq 2$.

THEOREM A. *Let Ω be a simply connected domain in $\mathbf{R}^2$ ($\Omega \neq \mathbf{R}^2$), let f be a conformal mapping from Ω onto the unit disk $|z|<1$, and let Γ be any line or circle on the plane. Then there exists an absolute constant p_0 ($1 < p_0 < 2$) such that*

$$\int_{\Gamma \cap \Omega} |f'(z)|^p \, |dz| \leq C(p, \Omega) < \infty \tag{0.1}$$

for $1 \leq p \leq p_0$.

For the development of the theorem, see [4], [5], [7], and [8]. Recently, Baernstein [1] constructed Ω, f, and Γ as in Theorem A, so that

$$\int_{\Gamma \cap \Omega} |f'(z)|^{2-\delta} \, |dz| = \infty$$

for some $\delta > 0$.

Suppose that G is the Green function on Ω with pole at $f^{-1}(0)$. It follows from (0.1) that

$$\int_{\Gamma \cap \Omega} |\nabla G(z)|^p \, |dz| \leq C(p, \Omega) \operatorname{dist}(0, f(\Gamma))^{-p}. \tag{0.2}$$

We extend (0.2) to the following.

THEOREM. *Suppose that Ω is a domain in $\mathbf{R}^m$ ($m \geq 2$) that satisfies the $(m-1)$-dimensional density condition ($(m-1)$DC). Let P be a fixed point in Ω, G the Green function of Ω with pole at P, and Γ an $(m-1)$-dimensional hyperplane with $P \notin \Gamma$. Then there exists a constant $p_0 > 1$ depending on the $(m-1)$DC constant, so that if $1 \leq p \leq p_0$ then*

$$\int_{\Gamma \cap \Omega} |\nabla G(x)|^p \, d\sigma(x) < B, \tag{0.3}$$

where $d\sigma$ is the $(m-1)$-dimensional measure on Γ and B is a constant depending on p, the $(m-1)$DC constant, $\operatorname{dist}(P, \partial\Omega)$, and $\operatorname{dist}(P, \Gamma)$.

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