Commutator Relations and Identities in Lattice-Ordered Groups

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To Roger C. Lyndon - In Memorium

1. Introduction

There are a few occasions when a relation between the generators of a group implies that it holds throughout the group. Two classical examples are:

(1) if the generators of a group commute, the group is Abelian; and

(2) if the commutator of any pair of generators of a group commutes with all the generators of that group, then the group is nilpotent class 2.

It is well known [and trivial—see the proof of Theorem A(i)] that (1) holds for arbitrary lattice-ordered groups. In this paper, we establish

THEOREM 1. (a) *There is a lattice-ordered group G generated by elements a and b such that $[a,b]$ is in the center of G but G is not nilpotent of any class.*

(b) *Any lattice-ordered group generated by elements a and b and satisfying $[a,b,a]=[a,b,b]=1$ must be metabelian.*

Part (a) confirms a belief stated in [6]. Indeed, we will show (Theorem B) that any finite set of commutators being central is not enough to guarantee that the resulting lattice-ordered group is nilpotent. (If the number of generators exceeds 2, the resulting lattice-ordered group need not even be metabelian.) Part (b) states that although the relations $[a,b,a]=[a,b,b]=1$ on the generators are not enough to ensure nilpotency of the entire lattice-ordered group, they do ensure that the metabelian law $[[x,y],[z,t]]=1$ holds for all x, y, z, and t belonging to the lattice-ordered group. So a commutator *identity* is in fact implied by the original commutator *relations,* albeit a weaker identity than that for groups. A generalization is given in Section 3.

Theorem 1 may also be viewed as follows:

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