Affine Variation Formulas and Affine Minimal Surfaces

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1. Introduction

In Section 2, we state the necessary preliminaries concerning affine hypersurfaces M^n in the $(n+1)$-dimensional standard affine space $\mathbf{A}^{n+1}$. In Section 3, we give examples of affine minimal surfaces in $\mathbf{A}^3$. In particular, we completely classify all affine minimal translation surfaces. Also, by giving a counterexample, we show that the affine Berstein problem (which is solved affirmatively for convex surfaces by Calabi [2]) has a negative solution in the nonconvex case. In Section 4 we obtain, also in the nonconvex case, that the affine minimal hypersurfaces M^n in $\mathbf{A}^{n+1}$ are those which have an extremal volume under variations in the direction of the affine normal. Furthermore, we find that for nonconvex affine minimal surfaces, in contrast to the convex case, the second variation of the affine area does not necessarily have a sign.

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2. Preliminaries

Let $\mathbf{A}^{n+1}$ be the standard $(n+1)$-dimensional real affine space, that is, $\mathbf{R}^{n+1}$ endowed with the standard linear connection D and the volume element Ω given by the determinant. Then $D\Omega = 0$, and so (D, Ω) defines an *equi-affine structure* on $\mathbf{R}^{n+1}$ [4; 5].

Let M^n be an oriented hypersurface in $\mathbf{A}^{n+1}$. Then the natural problem of how to induce an equi-affine structure (∇, θ) on M^n starting from (D, Ω) on $\mathbf{A}^{n+1}$ was solved by Nomizu in the following way [4]. Let ξ be any *transversal vector field* on M^n such that, $\forall x \in M^n$, $T_x\mathbf{A}^{n+1} = T_xM^n \oplus \operatorname{span}\{\xi_x\}$. Consider the corresponding formulas of Gauss and Weingarten:

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \qquad D_X \xi = -SX + \tau(X)\xi,$$

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