Direct Limits in an Equivariant K Theory Defined by Proper Cocycles

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Having observed the important role played in a variety of settings by the K theory of the reduced group C^* algebras of infinite discrete groups, Baum and Connes [2] considered the K theory of the reduced crossed product C^* algebras arising from smooth actions of Lie groups on smooth manifolds. The fundamental insights of [2] are that for a group G and a manifold X as above: one can use as cocycles manifolds on which G acts smoothly and properly to define groups $K^*(G,X)$; it is reasonable to conjecture that $K^*(G,X)$ is isomorphic to $K_*(C^*(G, C_0(X)))$ via an index map; and if this conjecture is true, then so are many other interesting conjectures. (Because all crossed products are reduced in this paper, the subscript r will be omitted from the notation.)

The definition of $K^*(G,X)$ using vector bundles with finite-dimensional fibers that was suggested in [2] was replaced in [3] by a slightly different definition. Because it may be that neither [2] nor [3] is available to the reader and because both are sketchy in their treatment of details, the first section of the present paper is a full discussion of the definition given in [3]. A remark at the end of the paper discusses the role of finite-dimensional vector bundles.

Our main result is a theorem concerning the behavior of $K^*(G,X)$ under certain direct limits, of the type $G=\varinjlim_i G_i$, that is analogous to a theorem about $K_*(C^*(G, C_0(X)))$. Not only does the analogy between these theorems support the Baum–Connes conjecture, but it can be used to prove the conjecture for any Lie group which is the direct limit of open and closed compact subgroups (e.g., a discrete group in which each finitely generated subgroup is finite) and for any abelian Lie group. The main theorem is stated and proved in Section 2, where the above-mentioned cases of the conjecture are proved as corollaries. The proof of the main theorem is interesting in that it illustrates the role that the classifying spaces introduced in [12] and [13] play in the K theory of transformation group C^* algebras.

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