A Converse Fatou Theorem

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1. Introduction

A well-known result of Fatou states that every positive solution of the Laplace equation on the upper half-space $\mathbf{R}_+^{n+1}$ has a finite nontangential limit at Lebesgue almost every point of the horizontal boundary $\mathbf{R}^n$ (cf. [2], [9]). It is also known that the nontangential approach region (a cone) in this result cannot be replaced by one which is bounded by a surface tangential to the boundary (cf. [4], [11]).

However, there are many surfaces which do not lie entirely in a cone, yet are not tangential to the boundary. Recently, Nagel and Stein [6] obtained a new Fatou theorem for solutions of the Laplace equation on $\mathbf{R}_+^{n+1}$. Their approach regions allow sequential approach to the boundary at any desired degree of tangency. Their results were generalized in [5] to improve the classical approach regions for certain parabolic equations on $\mathbf{R}_+^{n+1}$ and for the heat equation on the right half-space.

The most important condition on these new approach regions Ω involves the Lebesgue measure of their cross sections

$$\Omega(t) = \{x \in \mathbf{R}^n : (x, t) \in \Omega\}$$

for every height $t > 0$. However, it is intuitively clear that boundary limits from within an approach region only involve the structure of the region close to the boundary point. It is shown in Section 2 that this is indeed true. There we obtain a Fatou theorem for "locally admissible" regions.

In Section 3 we show that these locally admissible regions are the only ones which permit every bounded solution to have finite limits from within them at almost every boundary point. The proof of this result is accomplished by reducing to the case of a sequence of convolution operators on a group with finite Haar measure and then applying techniques developed by Stein [8] and Sawyer [7]. It appears that this reduction to the case of finite measure could be avoided by invoking Stein's theorem as presented in Chapter 6 of [3].

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