COMPLETE SPECTRAL AREA ESTIMATES AND SELF-COMMUTATORS

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Dedicated to Professor Shōzō Koshi on his sixtieth birthday

1. Introduction. Let A be a uniform algebra on a compact Hausdorff space X; that is, A is a closed subalgebra of $C(X)$ which contains the constant functions and which separates the points of X. The spectrum $\sigma(f)$ of $f \in A$ is the compact set of complex numbers λ such that $1/(\lambda - f)$ does not belong to A. The norm $\|u\|$ of $u \in C(X)$ is the supremum on X of the absolute value of u. Alexander [2, Lemma 2] proved the following theorem, using a quantitative version of the classical Hartogs–Rosenthal theorem on rational approximation in the complex plane.

ALEXANDER'S THEOREM. *If f is in A then*

$$\operatorname{dist}(\bar{f}, A) \leq \{\operatorname{Area}(\sigma(f))/\pi\}^{1/2},$$

where $\bar{f}$ is the complex conjugate of f and $\operatorname{dist}(\bar{f}, A) = \inf\{\|\bar{f} - g\| : g \in A\}$.

In Section 3 of this paper we give a new proof of this theorem. The proof we give is very abstract. We use a distance formula in a uniform algebra, which will be proved in Section 2, and we will need the famous Putnam inequality in operator theory. In Section 4, using Alexander's theorem, we will give an area estimate of a complete spectral set for the distance from the adjoint T^* of T to some norm closed algebra generated by T and $(T-\lambda)^{-1}$, where T is a bounded linear operator and λ is not in the complete spectral set. In Section 5, we will give an area estimate of the spectrum $\sigma(T)$ of a hyponormal operator T for the distance from T^* to some weakly closed algebra generated by T and $(T-\lambda)^{-1}$ for $\lambda \notin \sigma(T)$. In Section 6, we will show an area estimate of the complete spectrum set for the self-commutator of a bounded linear operator. This estimate looks like the Putnam inequality.

In this paper $\mathcal{H}$ denotes a Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. If T is in $\mathcal{L}(\mathcal{H})$ and $T^*T - TT^*$ is nonnegative then we call T a *hyponormal* operator. In Section 6 we shall show that, if T is a hyponormal operator and K is in $\mathcal{L}(\mathcal{H})$ with $KT = TK$, then

$$\|T^*K - KT^*\| \leq 2\{\operatorname{Area}(\sigma(T)\}^{1/2}\|K\|.$$

Moreover, for any T in $\mathcal{L}(\mathcal{H})$ we shall show the result above with a complete spectral set instead of $\sigma(T)$.

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