TRANSITIVE GROUP ACTIONS AND RICCI CURVATURE PROPERTIES

Isabel Dotti Miatello

1. Introduction. When considering left invariant metrics on Lie groups, Milnor [11] discusses the problem of determining the real Lie groups admitting left invariant metrics and satisfying certain curvature restrictions. For example, it is shown in [11] that if for some metric the Ricci curvature is nonnegative then the group must be unimodular, and if the Ricci curvature is positive then the group must be compact and semi-simple.

We generalize in Proposition 2 and Corollary 2 the above results for homogeneous M. On the other hand, we prove in Theorem 1 a splitting result related to transitive subgroups of $I(M)$, where M is a homogeneous manifold of nonnegative Ricci curvature, and derive as a corollary that if the subgroup is solvable then the manifold must be flat; if the Ricci curvature is positive there exists a compact and semi-simple Lie group acting transitively by isometries.

Related to negatively Ricci curved homogeneous manifolds, it is not yet known in general whether any $M=G/H$ does admit an invariant metric with $\mathrm{Ric}<0$, even in the simply transitive case. In Theorem 2 we find a strong restriction on the transitive unimodular Lie subgroups of the full isometry group: they are semi-simple (noncompact). In particular, from results of Gordon [9], the full isometry group is semi-simple.

2. Ricci curvature of Riemannian submersions. Let $P \xrightarrow{\pi} M$ be a Riemannian submersion; that is, for x in M there is an orthogonal splitting $T_x P = V_x \oplus H_x$ into vertical and horizontal subspaces, and $H_x \xrightarrow{\pi_*} T_{\pi(x)} M$ is an isometry for all x.

Let $\mathcal{H}$ and $\mathcal{V}$ denote (respectively) the orthogonal projections onto the horizontal and vertical subspaces of $T_x P$ at any point, and set (see [13])

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E} F + \mathcal{H}\nabla_{\mathcal{V}F} E$$

for E and F vector fields on P.

The following lemma is a consequence of O'Neill's identities for the curvature of a Riemannian submersion. See also Chapter 9 of [4].

LEMMA 1. *Let $P \xrightarrow{\pi} M$ be a Riemannian submersion with totally geodesic fibers, and let X and Y be horizontal unit vectors at x in P. Then*

$$\mathrm{Ric}(\pi_* X, \pi_* Y) = \mathrm{Ric}(X, Y) + 2\sum_i \langle A_{H_i} X, A_{H_i} Y \rangle,$$

where $\{H_i\}$ is an orthonormal local basis of horizontal vector fields.

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