STRONG LAWS OF LARGE NUMBERS FOR WEAKLY CORRELATED RANDOM VARIABLES

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Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of complex-valued random variables on a probability space (Ω, P) such that

$$\|X_n\|^2 = E[|X_n|^2] = \int_\Omega |X_n(\omega)|^2 \, dP(\omega) \le 1. \tag{1}$$

We are interested primarily in second-order conditions assuring the strong law of large numbers

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} X_n = 0 \text{ a.s.} \tag{SLNN}$$

The simplest case concerns uniformly bounded random variables.

THEOREM 1. *Let* $|X_n| \le 1$ *a.s. and suppose that*

$$\sum_{N\ge 1} \frac{1}{N} \left\| \frac{1}{N} \sum_{n\le N} X_n \right\|^2 < \infty. \tag{2}$$

Then the SLLN *holds.*

This theorem is essentially known, various special cases having been used in [2], [1], [10, p. 31], [9, §§III.4, IV.4]. While [8] presents almost as general a theorem, apparently Theorem 1 has not appeared explicitly in print. The proof of this and our other theorems consists in showing that the SLLN holds along some subsequence $\{N_k\}$ and then applying a suitable maximal inequality to interpolate between the N_k. When the random variables are uniformly bounded, the maximal inequality is trivial. The heart of Theorem 1, then, is the following refinement of the principle of Cauchy condensation.

LEMMA 2 [2]. *Let* $\{a_n\}_{n=1}^{\infty}$ *be real numbers such that*

$$a_n \ge 0, \qquad \sum_{n\ge 1} \frac{a_n}{n} < \infty. \tag{3}$$

Then there exists an increasing sequence of integers $\{n_k\}$ *such that* $\sum_{k\ge 1} a_{n_k} < \infty$ *and* $n_{k+1}/n_k \to 1$.

We shall constantly use the following easy and well-known lemma.

LEMMA 3. *If* Y_n *are random variables such that* $\sum_{n\ge 1} \|Y_n\|^2 < \infty$, *then* $Y_n \to 0$ *a.s.*

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