# INTEGRAL DOMAINS IN WHICH EACH $t$-IDEAL IS DIVISORIAL

Evan Houston and Muhammad Zafrullah

**Introduction.** Let $D$ be a commutative integral domain, and let $\mathfrak{F}(D)$ be the set of nonzero fractional ideals of $D$. We recall the $v$- and $t$-operations. For each $A \in \mathfrak{F}(D)$, $A_v = (A^{-1})^{-1}$ = the intersection of the principal fractional ideals of $D$ which contain $A$, and $A_t = \bigcup \{J_v : J$ is a finitely generated subideal of $A\}$. If $A = A_v$ (resp. $A = A_t$) then $A$ is said to be a $v$-ideal or *divisorial* (resp. a *t-ideal*). The $v$-ideal $A$ has *finite type* if $A = J_v$ for some finitely generated $J \in \mathfrak{F}(D)$. The fractional ideal $A$ is *quasi-finite* if $A^{-1} = J^{-1}$ for some finitely generated fractional subideal $J$ of $A$. The $v$- and $t$-operations are examples of star operations; the reader is referred to [7, §§32, 34] or to [15] for the properties of star operations (which we shall use freely).

We define a domain $D$ to be a *TV-domain* if each $t$-ideal of $D$ is divisorial (equivalently, if the $v$- and $t$-operations on $D$ are the same). Our study of $TV$-domains is motivated by [12], where Heinzer studies domains *all* of whose nonzero ideals are divisorial. The class of $TV$-domains is, of course, much larger. It includes the class of *Mori domains,* domains satisfying the ascending chain condition on divisorial ideals. Hence Noetherian domains and Krull domains are $TV$-domains.

In the first section, we study the properties of $TV$-domains, generalizing many (but not all) of the results of [12]. Before describing the main result, we recall that each $t$-ideal $A$ of $D$ is contained in a $t$-ideal $M$ maximal among $t$-ideals containing $A$, and this *maximal t-ideal* is prime [15, pp. 30–31]. We prove (Theorem 1.3) that every (proper) $t$-ideal of a $TV$-domain $D$ is contained in only finitely many maximal $t$-ideals. We also show that every maximal $t$-ideal of a $TV$-domain $D$ has the form $(a):b$ for some $a, b \in D$.

In the second section we give several characterizations of Krull domains. In particular, we give a proof more direct than (say) that given in [5, pp. 12–16] of the fact that (a) a completely integrally closed Mori domain is a Krull domain. In fact we prove that (a′) a completely integrally closed $TV$-domain is a Krull domain. This is done as follows. First, define a fractional ideal $A$ of $D$ to be *v-invertible* (resp. *t-invertible*) if $(AA^{-1})_v = D$ (resp. $(AA^{-1})_t = D$). In [18, Thm. 0] it was proved, as a consequence of (a), that (b) $D$ is a Krull domain if and only if every $A \in \mathfrak{F}(D)$ is $t$-invertible. Here we prove (b) independently of (a) and then use (b) to prove (a′), yielding a more satisfying approach to a well-known result.

The third section characterizes Prüfer $v$-multiplication domains among $TV$-domains, and we show that this class of rings is a certain subclass of the class of

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