NUMERICAL RADII OF ZERO-ONE MATRICES

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1. Introduction. Recently V. Müller (see [6]) has constructed a striking example related to the following long-standing problem: What is the best constant C such that $w(TS) \leq Cw(T)\|S\|$ for all commuting operators T and S? The setting here is complex Hilbert space and $\|S\|$ denotes the usual operator norm. The numerical radius $w(T)$ may be defined as $\sup\{|(Tu,u)|: \|u\|=1\}$. Müller's example shows that $C > 1.01$, in spite of various results that appeared to support the conjecture $C=1$ (for further details see §3).

A study of Müller's example led us to related operators that have the following advantages. They are simpler to describe and can be set in a lower-dimensional space (dimension 9 seems to be the best we can do; the example of Müller lives in a 12-dimensional space). The computations can be carried out directly and the relevant numerical radii identified (Müller relies on a computer check of some aspects of his example). The constant C is more closely constrained by our examples (we shall see that $C \geq 1/\cos(\pi/9) > 1.064$; the best upper bound known for C appears to be $C < 1.169$). Müller refers to an approximation result of Holbrook (see [5]) to establish that $w(TS)$ may exceed $w(T)\|S\|$ even when T is a polynomial in S; in some of our examples this feature is built in. Finally, we can adapt our examples to settle a related problem about ρ-dilations (see §3).

We end this introduction by describing explicitly one of our examples. Let $S = S_9$, the shift on the Hilbert space of dimension 9. Let $T = S^3 + S^7$. Then, of course, $\|S\| = 1$, and we shall see that $w(T) = \cos(\pi/10)$ while $w(TS) = 1$, so that $C \geq 1/\cos(\pi/10)$ (>1.05).

2. Zero-one matrices. Given an $n \times n$ matrix M whose entries are zeroes or ones, we form the incidence graph $G(M)$ with vertices labelled 1 through n and edges joining exactly those pairs (i,j) such that $m_{ij}=1$.

PROPOSITION 1. *Suppose that an $n \times n$ zero-one matrix M has zero diagonal and, for each $1 \leq k \leq n$, no more than two ones in the cross-shaped region X_k defined as the union of the kth row and the kth column. Then*

$$w(M) = \cos(\pi/(L+1)),$$

where L is the number of vertices in the longest chain in the incidence graph $G(M)$.

REMARKS. The condition on the X_k in the statement of our proposition is a way of saying that $G(M)$ has no vertices of valence greater than 2, that is, that no more than two edges meet at a given vertex. As a result, $G(M)$ breaks into

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