A NEWLANDER-NIRENBERG THEOREM FOR MANIFOLDS WITH BOUNDARY

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The Newlander–Nirenberg theorem [7] states that if M is a manifold with an integrable almost complex structure, then M is actually a complex manifold. Thus, about any $z_0 \in M$ there exist coordinate functions $z_1, \dots, z_n$ such that the almost complex structure defined by $z_1, \dots, z_n$ coincides with the given almost complex structure on M. If $\bar{M}$ is a manifold with smooth boundary such that the almost complex structure extends smoothly to the boundary, then it is natural to ask if the assumption of integrability still implies the conclusions of the Newlander–Nirenberg theorem. Hill [4] has constructed counterexamples showing that such a theorem does not hold in general. In this paper we show that the analog of the Newlander–Nirenberg theorem does hold if the boundary of $\bar{M}$ is pseudoconvex near z_0.

We now state precisely what we mean by an almost complex structure that extends smoothly to the boundary. Suppose that $\bar{M}$ is a manifold of real dimension $2n$ with smooth boundary, and let U be a neighborhood in the relative topology of $\bar{M}$ of a given boundary point z_0. We shall say that an *almost complex structure* is defined in U if there exists a subbundle $\mathfrak{L}$ of fiber dimension n of the complexified tangent bundle $\mathbf{C}T(\bar{M})$ such that for each $z \in U$, $\mathfrak{L}_z \cap \bar{\mathfrak{L}}_z = 0$. The structure is said to be *integrable* if $\mathfrak{L}$ is closed under brackets; that is, if L' and L'' are arbitrary sections of $\mathfrak{L}$, then $[L', L'']$ is again a section of $\mathfrak{L}$.

Observe that if $\bar{N}$ is a smoothly bounded complex manifold, then the bundle $T^{1,0}$ of holomorphic tangent vectors defines an integrable almost complex structure which is called the complex structure of $\bar{N}$. To show that $\bar{M}$ possesses a complex structure, it suffices to construct, in a neighborhood U of each point $z_0 \in \bar{M}$, a set of smooth functions $f_1, \dots, f_n$ with linearly independent differentials such that each function f_j is "holomorphic" with respect to the almost complex structure, that is, such that $\bar{L} f_j \equiv 0$ for every section L of $\mathfrak{L}$. In fact, if we view $f = (f_1, \dots, f_n)$ as a coordinate map into $\mathbf{C}^n$, then the bundle $\mathfrak{L}$ satisfies $f_* \mathfrak{L} = T^{1,0}$, which is the complex structure of $\mathbf{C}^n$. Thus near z_0 we may view $\bar{M}$ as a complex manifold with smooth boundary.

On an integrable almost-complex manifold, the usual $\bar{\partial}$-formalism carries through with no changes. If $\mathfrak{L}_z^p$ denotes the p-fold product $\mathfrak{L}_z \oplus \cdots \oplus \mathfrak{L}_z$, then $\Lambda_z^{p,q}$ is the space of alternating tensors on $\mathfrak{L}_z^p \oplus \bar{\mathfrak{L}}_z^q$. Because of the integrability assumption, it follows that if w is a section of $\Lambda^{p,q}$ then dw can be written as a sum $\partial w + \bar{\partial} w$, where $\partial w \in \Lambda^{p+1,q}$ and $\bar{\partial} w \in \Lambda^{p,q+1}$. From this one also obtains the familiar identity $\bar{\partial} \circ \bar{\partial} = 0$.

Now suppose that $r(z)$ is a boundary-defining function for $\bar{M}$. This means that $r < 0$ on M, $bM = \{z \in \bar{M};\, r(z) = 0\}$, and $dr(z) \neq 0$ when $z \in bM$. We say that bM

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