GROUP ACTIONS ON TREES AND LENGTH FUNCTIONS

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Introduction. A group of isometries acting on a metric tree gives a length function in the sense of Lyndon [6], associated with each point of the tree. In this paper certain properties of length functions are considered, in particular the existence of extensions, and relationships with corresponding properties of group actions are described.

The quotient of a group action on a metric space is described in Section 1. Group actions on trees and relationships between non-Archimedean elements, bounded actions, and fixed points are considered in Sections 2 and 3. These results are used in Section 4 to relate extensions of length functions to actions of factor groups on quotient trees, in Theorems 4.2 and 4.3. Results on extensions of length functions can therefore be translated to group actions on trees. An example is given in Theorem 4.5, where the action of any hypercentral group is described.

1. Factors of actions on metric spaces. Let a group K act as a group of isometries on a metric space X, equipped with a metric d. Define a relation between the elements of X by $u \sim v$ if $\inf_{x \in K} d(u, xv) = 0$.

LEMMA 1.1. *The relation* $\sim$ *is an equivalence relation on* X.

Proof. The relation is clearly reflexive, and since (for any $x \in K$) $d(u, xv) = d(xv, u) = d(v, x^{-1}u)$, it is symmetric.

For any $x, y \in K$ and $u, v, w \in X$, by the triangle inequality

$$d(u, xyw) \le d(u, xv) + d(xv, xyw) = d(u, xv) + d(v, yw).$$

Thus, if $u \sim v$ and $v \sim w$ then $u \sim w$ and the relation is transitive. □

Let X/K be the set of equivalence classes under $\sim$, and denote the equivalence class of u by $[u]$.

PROPOSITION 1.2. *X/K is a metric space with metric d' defined by*

$$d'([u], [v]) = \inf_{x \in K} d(u, xv).$$

Moreover, if X is complete then so is X/K.

Proof. We first show that d' is well defined. If $u' \sim u$, $v' \sim v$, then for $x, y, z \in K$

$$d(yu', xzv') \le d(yu', u) + d(u, xv) + d(xv, xzv');$$

that is,

$$d(u', y^{-1}xzv') \le d(u, yu') + d(u, xv) + d(v, zv').$$

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