OPERATORS WITH COMMUTATIVE COMMUTANTS

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It is a well-known consequence of the Putnam–Fuglede theorem that if a normal operator N is a quasiaffine transform of a normal operator M, then M and N are unitarily equivalent. Problem 199 of [4] shows that the result does not remain true if M and N are merely subnormal, even if they are quasisimilar. In the present paper we show that if M (resp. N) is the direct sum of k (resp. m) copies of an operator A having a commutative commutant, where m and k are countable cardinalities, and if N is a quasiaffine transform of M, then $k=m$ (see Theorem 1). In the special case where A is the simple unilateral shift, this extends a result of Hoover [5], who shows that quasisimilar isometries are unitarily equivalent. In fact, using a result of Fan [3], we show that if an isometry N is a quasiaffine transform of an isometry M, and if the unitary part of the Wold decomposition of N has a singular scalar-valued spectral measure, then M and N are unitarily equivalent (see Theorem 2).

We conclude the paper with a result about multiplications M_z by $g(z)\equiv z$ on function spaces $R^2(X,\mu)$, where μ is a positive measure supported on a compact subset X of $\mathbf{C}$; every nonscalar operator commuting with $(M_z)^{(n)}$ has a hyperinvariant subspace if $(M_z)^*$ has an eigenvalue and $n<\infty$ (see Theorem 3). This generalizes a result of Sz.-Nagy and Foiaş [8, p. 191] and Nordgren [6] about the unilateral shift (see also [7, p. 149]).

Let us here fix some notations and definitions. For the commutant of an operator A we use the usual notation $\{A\}'$. If A is an operator on a Hilbert space $\mathcal{H}$, then $A^{(k)}$ denotes the direct sum of k copies of A acting on the direct sum $\mathcal{H}^{(k)}$ of k copies of $\mathcal{H}$, where k is any cardinality; if $k=0$ then $\mathcal{H}^{(0)}=\{0\}$. A bounded linear transformation between two Banach spaces is called a quasiaffinity if it is injective and has dense range; an operator N is a quasiaffine transform of an operator M if $CM=NC$ for some quasiaffinity C. The operators M and N are quasisimilar if $C_1M=NC_1$ and $MC_2=C_2N$ for some quasiaffinities C_1 and C_2.

For a compact subset X of $\mathbf{C}$, $\mathrm{Rat}(X)$ denotes the algebra of all rational functions with poles off X. If μ is a positive Borel measure supported on X, then $R^2(X,\mu)$ denotes the closure of $\mathrm{Rat}(X)$ in $\mathcal{L}^2(\mu)$.

An operator A on $\mathcal{H}$, with spectrum contained in X, is called $\mathrm{Rat}(X)$-cyclic if there exists a vector e in $\mathcal{H}$ such that the linear manifold $\{r(A)e : r\in \mathrm{Rat}\, X\}$ is dense in $\mathcal{H}$.

THEOREM 1. *Let $A\in B(\mathcal{H})$ and assume $\{A\}'$ is commutative. Let C be a bounded linear transformation such that $CA^{(k)}=A^{(m)}C$ for some finite or countable cardinalities k and m. Then*

(a) *$k\le m$ if C is injective, and*

(b) *$k\ge m$ if C has dense range.*

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