A SHARP GOOD-λ INEQUALITY WITH AN APPLICATION TO RIESZ TRANSFORMS

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0. Introduction. Let $f \in \mathcal{S}(\mathbf{R}^n)$, the class of rapidly decreasing functions in $\mathbf{R}^n$. For $j = 1, 2, \ldots, n$, define the Riesz transforms as the multiplier operators $(R_j f)\hat{\ }(\xi) = (i\xi_j/|\xi|)\hat{f}(\xi)$ and set

$$Rf(x) = \left(\sum_{j=1}^{n} |R_j f(x)|^2\right)^{1/2}$$

In [22] Stein proved that $\|Rf\|_p \le C_p \|f\|_p$ for $1 < p < \infty$, with C_p independent of the dimension n. Alternative probabilistic proofs are given in [1], [4], [14], and [16], and in [9] the result is proved by using the method of rotations. Whereas all of these proofs give constants independent of n, none of them give the correct behavior with respect to p (see the remarks following Corollary 2.3). On the other hand, the classical proof [20] which gives constants depending on the dimension does give the right asymptotic behavior with respect to p. That is, C_p is $O(p)$ as $p \to \infty$ and $O(1/(p-1))$ as $p \to 1$. It seems unnatural to us that this should be lost when passing to constants independent of n. The purpose of this note is to correct this deficiency. We do this by proving a sharp good-λ inequality (Lemma 1.2) for vector-valued martingales which itself may be of independent interest.

1. The good-λ inequality. Throughout the paper, we use the following notation. If X_t is an L^p-bounded martingale with $1 < p < \infty$, X will denote the random variable in L^p such that $X_t = E(X \mid \mathcal{F}_t)$ and $\langle X \rangle_t$ will denote the square function of X_t. By $\langle X \rangle$ we shall mean $\langle X \rangle_\infty$. All our martingales are on the Brownian filtration and hence always continuous.

THEOREM 1.1. *Let $\{X^i\}_{i=1}^m$ be random variables in L^p with $2 \le p < \infty$ and such that $EX^i = 0$ for all i. There are universal constants C_1 and C_2 (independent of p and m) such that*

$$\left\| \left(\sum_{i=1}^{m} |X^i|^2 \right)^{1/2} \right\|_p \le C_1 \sqrt{p} \left\| \left(\sum_{i=1}^{m} \langle X^i \rangle \right)^{1/2} \right\|_p, \tag{1.1}$$

$$\left\| \left(\sum_{i=1}^{m} \langle X^i \rangle \right)^{1/2} \right\|_p \le C_2 \sqrt{p} \left\| \left(\sum_{i=1}^{m} |X^i|^2 \right)^{1/2} \right\|_p. \tag{1.2}$$

Part (1.2) of the theorem is well known but we shall prove it here for the convenience of the reader. However, for (1.1) all of the proofs known to the author will give, at best, constants of order p. To prove (1.2) we recall the following lemma.

Received May 12, 1987.
The author was supported by an NSF Postdoctoral Fellowship.
Michigan Math. J. 35 (1988).