INVARIANTS OF COMPLEX FOLIATIONS AND THE MONGE–AMPÈRE EQUATION

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0. Introduction. In this paper we study the local and global geometric invariants associated to a foliation of a complex manifold by complex submanifolds. We call such foliations *complex foliations* and do not require them to be transversely holomorphic (i.e., the leaves of the foliation might not fit together in a holomorphic way)—indeed, one of the objects of the present paper is to determine global conditions which force complex foliations to be transversely holomorphic.

Our primary motivation for undertaking such a study comes from the complex Monge–Ampère equation. In [2] it is observed that a smooth, real-valued function u on a complex n-dimensional manifold M, satisfying the equations $(\partial\bar{\partial}u)^{p+1}=0$ and $(\partial\bar{\partial}u)^{p}\neq 0$, gives rise to a foliation of M by complex submanifolds of complex codimension p, a Monge–Ampère foliation, in the following way. The closed $(1,1)$-form $\partial\bar{\partial}u$ defines a distribution $L=\{X\in TM \mid i(X)\partial\bar{\partial}u=0\}$, which is easily shown to be integrable.

The technique of exploiting the geometry of Monge–Ampère foliations to study the Monge–Ampère equation has been used a great deal in recent years ([1], [5], [10], [11], and [13]). We call particular attention to the paper of Lempert, where the Kobayashi metric is used to associate a canonical complex foliation, singular at one point, to a smooth, strongly convex domain in $\mathbb{C}^n$. The results of [3] show that the geometry of the entire domain can be recovered from the foliation germ at the singularity together with metric data on the leaves.

In most of the above work the solution of the Monge–Ampère equation is used heavily in the analysis of the associated foliation, but there are interesting examples of foliations which are not Monge–Ampère (e.g., those arising in twistor theory [6]), and even when a foliation does arise from a solution of the Monge–Ampère equation, the solution may not be known but the geometry of the foliation itself may be of interest. For example, it would be interesting to have a characterization of which germs of singular foliations can arise from the Lempert foliation of a convex domain. A systematic study of complex foliations is therefore of interest, and this paper initiates such a study.

The outline of the paper is as follows. Our notational conventions, as well as some basic definitions and useful local formulas, are presented in Section 1.

In Section 2 the relative de Rham complex $(\Omega_{\mathfrak{F}}^{k}, d_{\mathfrak{F}})$ of a complex foliation is presented. Here $\Omega_{\mathfrak{F}}^{k}$ denotes the sheaf of sections of the bundle $\wedge^{k}L^{*}$, where L is the tangent bundle of $\mathfrak{F}$ and the operator $d_{\mathfrak{F}}$ is exterior differentiation in the

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