GREEN'S THEOREM AND BALAYAGE

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1. Introduction. For Q a cube in $\mathbf{R}^d$ with sides parallel to the coordinate axes, let $|Q|$ denote Q's Lebesgue measure. For $\phi \in L^1_{\text{loc}}(\mathbf{R}^d)$, define

$$\phi_Q \equiv \frac{1}{|Q|} \int_Q \phi \, dx.$$

We say that ϕ is in BMO if

$$\|\phi\|_* \equiv \sup_{Q \subset \mathbf{R}^d} \frac{1}{|Q|} \int_Q |\phi - \phi_Q| \, dx < \infty.$$

For Q as above we set $\hat{Q} = \{(t, y) \in \mathbf{R}^{d+1}_+ : t \in Q,\ 0 < y < \ell(Q)\}$, where $\ell(Q)$ is the sidelength of Q. We say that μ, a Borel measure on $\mathbf{R}^{d+1}_+$, is a *Carleson measure* if

$$\|\mu\|_C \equiv \sup_{Q \subset \mathbf{R}^d} \frac{|\mu|(\hat{Q})}{|Q|} < \infty.$$

There is an intimate connection between the space BMO and the family of Carleson measures. Roughly speaking, a Carleson measure is a conformally invariant finite measure, while a BMO function is a conformally invariant L^1 function. This connection is made more explicit through the following fact. Let $K \in L^1(\mathbf{R}^d)$ satisfy $\int K = 1$, $|K(x)| \le (1+|x|)^{-d-1}$, $|\nabla K(x)| \le (1+|x|)^{-d-2}$. For $y > 0$ let $K_y(x) = y^{-d} K(x/y)$. Consider the function

$$S_{\mu,K} \equiv \int_{\mathbf{R}^{d+1}_+} K_y(x-t) \, d\mu(t,y), \tag{1}$$

where μ is a Borel measure on $\mathbf{R}^{d+1}_+$. It is easy to see that if μ is finite then the integral in (1)—called the *sweep* or *balayage* of μ with respect to K—converges absolutely for a.e. $x \in \mathbf{R}^d$, and $\|S_{\mu,K}\|_1 \le C(d) \|\mu\|$.

More is true if μ is a Carleson measure. In that case, $S_{\mu,K} \in$ BMO and

$$\|S_{\mu,K}\|_* \le C(d) \|\mu\|_C. \tag{2}$$

The proof of (2) is quite easy. What is more remarkable (and also true) is that (2) has a converse [1; 2; 3].

THEOREM A. *Let $K \in L^1(\mathbf{R}^d)$ satisfy $\int K = 1$, $|K(x)| \le (1+|x|)^{-d-1}$. Let $\phi \in$ BMO have compact support. There exist $g \in L^\infty(\mathbf{R}^d)$ and a finite Carleson measure μ such that $\phi(x) = g(x) + S_{\mu,K}(x)$, where $\|g\|_\infty + \|\mu\|_C \le C(d) \|\phi\|_*$.*

The proofs in [1; 2; 3] work by an iteration argument. One builds a $\tilde{g}$ and a $\tilde{\mu}$ for which $\tilde{\phi} \equiv \tilde{g} + S_{\tilde{\mu},K}$ is close to ϕ in BMO, and then one repeats the argument on $\phi - \tilde{\phi}$. One does this infinitely often, obtaining g and μ in the limit. The effect

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