AN EMBEDDING THEOREM FOR THE FELL TOPOLOGY

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1. Introduction. Let 2^X (resp. $\mathrm{CL}(X)$) denote the closed (resp. closed and non-empty) subsets of a metric space $\langle X, d\rangle$. The fundamental topology on $\mathrm{CL}(X)$ is the *Hausdorff metric topology,* induced by the infinite-valued metric on $\mathrm{CL}(X)$ defined by

$$h_d(E,F)=\sup(\{d(x,E):x\in F\}\cup\{d(x,F):x\in E\}).$$

If we replace d by the metric $\rho=\min\{d,1\}$, then h_ρ is finite-valued and determines the same topology on $\mathrm{CL}(X)$. Most importantly [8], the map $E\to\rho(\cdot,E)$ is an isometry of $\langle \mathrm{CL}(X), h_\rho\rangle$ into the bounded continuous real functions on X, equipped with the usual uniform metric.

A somewhat weaker topology on 2^X, which agrees with the Hausdorff metric topology on $\mathrm{CL}(X)$ if and only if X is compact, is the *Fell topology* [10], also called the *topology of closed convergence* [12]. To describe this topology, we introduce the following notation: if $A\subset X$, then

$$A^-=\{E\in 2^X : E\cap A\neq\varnothing\}\quad\text{and}\quad A^+=\{E\in 2^X: E\subset A\}.$$

The Fell topology τ_F has as a subbase all sets of the form V^-, where V is an open subset of X and $(K^C)^+$, where K is a compact subset of X. Obviously, the Fell topology is similar in spirit to the stronger *Vietoris topology* ([12], [14]). With respect to convergence notions, it is known (cf. [3, Lemma 1.0] or [11, p. 353]) that a sequence $\langle E_n\rangle$ in 2^X converges in the Fell topology to a closed set E if and only if

$$E=\mathrm{Li}\,E_n=\mathrm{Ls}\,E_n,$$

where $\mathrm{Li}\,E_n$ (resp. $\mathrm{Ls}\,E_n$) consists of all points x each neighborhood of which meets $\langle E_n\rangle$ eventually (resp. frequently). In the literature, this form of convergence is usually called *Kuratowski convergence,* but sometimes it goes by the name *topological convergence* [15]. In an arbitrary Hausdorff space (not necessarily metrizable), Kuratowski convergence is stronger than convergence with respect to the Fell topology; also, Kuratowski convergence of *nets* of sets determines the Fell topology if and only if X is locally compact (cf. [7] or [12]). The Fell topology and Kuratowski convergence of sets have been particularly important in the study of the convergence of lower semicontinuous functions and their minima ([1], [4], [5], [16]), but these notions also arise in probability theory ([13], [19]), mathematical economics [12], and the study of C^*-algebras [9].

In this note we show that a locally compact metrizable space X admits a metric d for which $E\to d(\cdot,E)$ is a topological embedding of $\langle \mathrm{CL}(X),\tau_F\rangle$ into the continuous real functions on X with the *compact-open topology* (*the topology of uniform convergence on compacta*), τ_{CO}, that can be extended to 2^X.

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