ON UNIFORM APPROXIMATION BY HARMONIC FUNCTIONS

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1. Introduction. Let X be a compact set in $\mathbf{R}^2$ and let $R(X)$ denote the uniform closure on X of functions analytic in a neighborhood of X. The following concept of analytic content defined by

$$\lambda(X) \stackrel{\text{def}}{=} \inf_{\phi \in R(X)} \|\bar{z}-\phi(z)\|_\infty \tag{1}$$

has been introduced in [7] and studied in [5], [7], [9], and [10] ($\|\cdot\|_\infty$ stands for the supremum norm in the space of continuous functions $C(X)$ on X). As easily follows from the Stone–Weierstrass theorem, $\lambda(X)=0$ if and only if $R(X)=C(X)$. This simple observation allows us to view $\lambda(X)$ as a certain measure of solvability of the problem of uniform approximation by analytic functions on X. As it turns out ([1] and [7]; see also [5], [9], and [10]), $\lambda(X)$ admits simple estimates in terms of basic geometric quantities of X such as area and perimeter. More precisely,

$$\left(\frac{A}{\pi}\right)^{1/2} \geq \lambda(X) \geq \frac{2A}{P}, \tag{2}$$

where $A=$ area of X, $P=$ perimeter of X (if X has a finite perimeter; otherwise, $P=\infty$); see [3, Ch. IV] and [8]. We mention that the inequality in the left-hand side of (2) was observed by Alexander [1] and the second inequality is due to the author [7]. We refer the reader to [5], [9], and [10] for a detailed discussion of these inequalities and related isoperimetric problems.

The purpose of this note is to develop a similar concept for $H(X)$, the uniform closure of the space of functions harmonic in a neighborhood of a compact set $X \subset \mathbf{R}^n$, $n \geq 2$. By similarity with (1) we define the harmonic content $\Lambda(X)$ of a compact set $X \subset \mathbf{R}^n$ to be

$$\Lambda(X) \stackrel{\text{def}}{=} \inf_{u \in H(X)} \||x|^2-u\|_\infty, \tag{3}$$

where $|x|^2=\sum_{i=1}^n x_i^2$, $x=(x_1,\dots,x_n) \in \mathbf{R}^n$. The analogy with (1) can be seen if one observes the correspondence between $H(X)=$ uniform closure of the kernel of Δ on X and $R(X)=$ uniform closure of the kernel of $\partial/\partial\bar{z}$, $|x|^2 : \Delta(|x|^2) \equiv 2n \equiv \text{const} \neq 0$ in $\mathbf{R}^n$ and $\bar{z} : (\partial/\partial\bar{z})(\bar{z}) \equiv 1 \neq 0$ in $\mathbf{R}^2$. (We use the standard notation: Δ denotes the Laplacian $\sum_{i=1}^n (\partial^2/\partial x_i^2)$ and $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$, $z = x+iy$.) However, in this case the equivalence

$$\Lambda(X)=0 \Leftrightarrow H(X)=C(X) \tag{4}$$

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