THE RANGE INCLUSION PROBLEM FOR ELEMENTARY OPERATORS

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Dedicated to the memory of Constantin Apostol

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on a separable infinite-dimensional complex Hilbert space $\mathcal{H}$. Let $A=(A_1,\ldots,A_n)$ and $B=(B_1,\ldots,B_n)$ denote n-tuples of operators and let $R=R_{AB}$ denote the *elementary operator* on $\mathcal{L}(\mathcal{H})$ defined by

$$R(X)=\sum_{i=1}^{n} A_i X B_i.$$

Let $\mathcal{J}$ denote a 2-sided ideal of $\mathcal{L}(\mathcal{H})$ ($\mathcal{J}\neq\mathcal{L}(\mathcal{H})$). The purpose of this note is to draw attention to the range inclusion problem for elementary operators, which asks for a characterization of the structure of an elementary operator R_{AB} whose range is contained in $\mathcal{J}$,

$$\operatorname{Ran} R_{AB}\subset\mathcal{J}. \tag{1.1}$$

(We note that if $\mathcal{J}\neq\{0\}$, then it is impossible to achieve the identity $\operatorname{Ran} R_{AB}=\mathcal{J}$; this is because each nonzero ideal contains $\mathcal{F}$, the ideal of finite rank operators, and if $\mathcal{F}\subset\operatorname{Ran} R_{AB}$, then $\operatorname{Ran} R_{AB}=\mathcal{L}(\mathcal{H})$ [2, Thm. 2.3].)

It is easy to illustrate sufficient conditions for the range inclusion (1.1). If for each i, $A_i\in\mathcal{J}$ or $B_i\in\mathcal{J}$, then clearly (1.1) holds. This condition is not, however, necessary for range inclusion, as shown by the following.

EXAMPLE 1.1. For $1\le p\le\infty$, let $\mathcal{C}_p$ denote the Schatten p-ideal [5, p. 91]. Suppose $1<p,q<\infty$ with $1/p+1/q=1$, and let $A\in\mathcal{C}_p\backslash\mathcal{C}_1$ and $B\in\mathcal{C}_q\backslash\mathcal{C}_1$; then for every $X\in\mathcal{L}(\mathcal{H})$, $AXB\in\mathcal{C}_1$ [5, p. 92].

For $T\in\mathcal{L}(\mathcal{H})$, let $s(T)$ denote the sequence of *s-numbers* of T [5, p. 59]; in the case when T is compact, the s-numbers are the eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity. For an ideal $\mathcal{J}$, let J denote the *ideal set* of $\mathcal{J}$ (see, e.g., [3; 6; 7; 8]); thus $T\in\mathcal{J}$ if and only if $s(T)\in J$ [3; 8]. For example, if $\mathcal{J}=\mathcal{C}_p$, then $J=l_p$ $(1\le p\le\infty)$. The range inclusion (1.1) for $n=1$ has been characterized by Loebl and the author [3] as follows.

THEOREM 1.2 [3, Thm. 5.6]. *Let $A,B\in\mathcal{L}(\mathcal{H})$ and let $\mathcal{J}$ be an ideal of $\mathcal{L}(\mathcal{H})$. The* elementary multiplication *operator $S=S_{AB}$, defined by $S(X)=AXB$ ($X\in\mathcal{L}(\mathcal{H})$), satisfies* $\operatorname{Ran} S\subset\mathcal{J}$ *if and only if the product sequence $s(A)s(B)$ belongs to J.*

(Note that Example 1.1 follows from Theorem 1.2 and the fact that $l_p\cdot l_q\subset l_1$.)

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