NONLINEAR SOLUTIONS OF NEVANLINNA-PICK INTERPOLATION PROBLEMS

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1. Introduction. This article concerns a nonlinear extension of the classical theory of Nevanlinna–Pick interpolation. A matricial form of the classical Nevanlinna–Pick interpolation problem is as follows:

(NP) Given a collection $\{z_1, \dots, z_k\}$ of complex numbers with $|z_j| < 1$, a set of vectors $\{x_1, \dots, x_k\}$ in $\mathbf{C}^n$ and a set of vectors $\{y_1, \dots, y_k\}$ in $\mathbf{C}^m$, find all $m \times n$ matrix functions F analytic on the unit disk such that
(i) $\|F(z)\| \le 1$ for $|z| < 1$, and
(ii) $F(z_j)x_j = y_j$ for $1 \le j \le k$.

The classical result is that solutions F to (NP) exist if and only if the Pick matrix

$$\Lambda(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \left[\frac{x_j^* x_i - y_j^* y_i}{1 - \bar{z}_j z_i}\right]_{1 \le i, j \le k}$$

is positive semidefinite (see e.g. [4]). Various recipes exist then for constructing the solutions.

A dual version of (NP) has also been studied.

(NP)$_*$ Given a collection $\{w_1, \dots, w_{k'}\}$ of complex numbers with $|w_j| < 1$, a set of vectors $\{\xi_1, \dots, \xi_{k'}\}$ in $\mathbf{C}^m$ and a set of vectors $\{\eta_1, \dots, \eta_{k'}\}$ in $\mathbf{C}^n$, find all $m \times n$ matrix functions F analytic on the unit disk such that
(i) $\|F(z)\| \le 1$ for $|z| < 1$, and
(ii)$_*$ $\xi_j^* F(w_j) = \eta_j^*$ for $1 \le j \le k'$.

Note that F is a solution of (NP)$_*$ if and only if $F^*(z) = F(\bar{z})^*$ is a solution of a problem of the type (NP). Thus (NP)$_*$ has a solution if and only if the Pick matrix

$$\Lambda_*(\underline{\underline{w}}, \underline{\underline{\xi}}, \underline{\underline{\eta}}) = \left[\frac{\xi_j^* \xi_i - \eta_j^* \eta_i}{1 - \bar{w}_i w_j}\right]_{1 \le i, j \le k'}$$

is positive semidefinite. It is also possible to combine these. For simplicity we assume that $z_i \ne w_j$ for $1 < i \le k$ and $1 \le j \le k'$.

(NP)$\cap$(NP)$_*$ Find all matrix functions F which solve (NP) and (NP)$_*$ simultaneously.

The Pick matrix $\Lambda(\underline{\underline{z}}, \underline{\underline{x}}, \underline{\underline{y}}, \underline{\underline{w}}, \underline{\underline{\xi}}, \underline{\underline{\eta}})$ for this problem is more involved but can be computed; it is given in [5].

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