HILBERT FUNCTIONS AND SYMBOLIC POWERS

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1. Introduction. The following question of Cowsik [4] inspired much of this paper: if R is a regular local ring and $p \subset R$ is a prime ideal, then is $\bigoplus_{n \geq 0} p^{(n)}$ Noetherian? Here $p^{(n)} = p^n R_p \cap R$ is the nth symbolic power of p. Cowsik proved if $\dim R/p = 1$ then $\bigoplus p^{(n)}$ Noetherian implies that p is a set-theoretic complete intersection. One of the main cases is when R is 3-dimensional and $\operatorname{ht} p = 2$. Positive results were obtained in [5] and [8], but recently Roberts [29] gave a counterexample to the general question. One of the main results of this paper is to give a necessary and sufficient criterion (Theorem 3.1) for $\bigoplus_{n \geq 0} p^{(n)}$ to be Noetherian which is relatively simple to apply. Namely, we are able to show that, if R is a 3-dimensional regular local ring and p is a height-2 prime of R, then $\bigoplus_{n \geq 0} p^{(n)}$ is Noetherian if and only if there exist k, ℓ, elements $f \in p^{(k)}$, $g \in p^{(\ell)}$, and $x \notin p$ such that $\lambda(R/(f,g,x)) = ek\ell$, with $e = \lambda(R/(p,x))$. Here $\lambda(\ )$ denotes length.

The proof of this result requires an understanding of Hilbert functions of m-primary ideals in 2-dimensional Cohen–Macaulay (C–M for short) local rings. In general, if R, m is a d-dimensional local C–M ring and I is an m-primary ideal, then there is a polynomial $P_I(n)$ of the form

$$e_0\binom{n+d-1}{d} - e_1\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1} e_{d-1}\binom{n}{1} + (-1)^d e_d$$

such that, for $n \gg 0$, $P_I(n) = \lambda(R/I^n)$. We define $H_I(n) = \lambda(R/I^n)$ for every $n \geq 0$, and call H_I the Hilbert function of I and P_I the Hilbert polynomial of I. Not a great deal is known about the coefficients $e_0, \ldots, e_d$ of $P_I(n)$. However, see [10], [21], [28], and [30].

Of course, e_0 is called the multiplicity of I and can be computed as follows: if R/m is infinite and $x_1, \ldots, x_d$ is a minimal reduction of I, then $e_0 = \lambda(R/(x_1, \ldots, x_d))$. Northcott [22] showed that $\lambda(R/I) \geq e_0 - e_1$ always holds while Narita [21] showed that $e_2 \geq 0$. Recently Kubota [11] proved that if $\lambda(R/I) = e_0 - e_1$ and $\lambda(R/I^2) = e_0(d+1) - e_1 d$, then necessarily $e_2 = \cdots = e_d = 0$. For our theorem on symbolic powers we need to improve this theorem. We are able to show (Theorem 2.7) that if $\lambda(R/I) = e_0 - e_1$ then necessarily $e_2 = \cdots = e_d = 0$. (From this it follows that $P_I(n) = H_I(n)$ for all $n \geq 0$ and also that $I^2 = (x_1, \ldots, x_d)I$.)

The basic method of proof is essentially the same as in papers of Rees [26] and Kubota [11], but pushed slightly differently. The methods also apply to considering the difference $P_I(n) - H_I(n)$. Following the lead of Morales [20], we consider when $P_I(n) = H_I(n)$ for all $n \geq 1$ (Theorem 2.11) and when $P_I(n) \geq H_I(n)$ for all $n \geq 1$ (Proposition 2.12), and later apply these to the case where $\bigoplus_{n \geq 0} I^n/I^{n+1}$ is

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