LIFTING OF OPERATORS AND PRESCRIBED NUMBERS OF NEGATIVE SQUARES

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1. Introduction. The problem we are interested in can be formulated in full generality as follows.

Let $\mathcal{H}_i$ be Hilbert spaces and let $J_i \in \mathcal{L}(\mathcal{H}_i)$ be symmetries ($J \in \mathcal{L}(\mathcal{H})$ is a symmetry if $J = J^* = J^{-1}$), $i = 1, 2$. Consider also $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that the number of negative squares of the self-adjoint operator $J_1 - T^*J_2T$ is a given cardinal κ—see Section 2 for the terminology. (We denote this situation by writing $\kappa^-(J_1 - T^*J_2T) = \kappa$.) For other Hilbert spaces $\mathcal{H}_i'$ and symmetries $J_i' \in \mathcal{L}(\mathcal{H}_i')$, consider $\tilde{\mathcal{H}}_i = \mathcal{H}_i \oplus \mathcal{H}_i'$ and $\tilde{J}_i = J_i \oplus J_i'$, $i = 1, 2$. If $\tilde{\kappa}$ is another given cardinal, then

$$(*) \qquad \begin{cases} \text{Give a description of all operators } \tilde{T} \in \mathcal{L}(\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2) \text{ such that} \\ P^{\tilde{\mathcal{H}}_2}_{\mathcal{H}_2} \tilde{T} \mid \mathcal{H}_1 = T, \quad \text{and} \quad \kappa^-(\tilde{J}_1 - \tilde{T}^*\tilde{J}_2\tilde{T}) = \tilde{\kappa}. \end{cases}$$

(For a closed subspace $\mathcal{G}$ of a Hilbert space $\mathcal{H}$, $P^{\mathcal{H}}_{\mathcal{G}}$ stands for the orthogonal projection onto $\mathcal{G}$.)

The "definite" case of Problem (*) (i.e., all the symmetries involved equal the identity and $\kappa = \tilde{\kappa} = 0$, making T and $\tilde{T}$ contractions) is a well-known problem in dilation theory. A full solution of it (which includes the description of the defect spaces of $\tilde{T}$) and its (long) history can be found in [3]. The methods involved proved to be useful for the geometric approach to dilation theory and to some extrapolation problems.

The passing to the "indefinite" case has strong motivations and many efforts have been made along this line both in extrapolation problems and in dilation theory (see, as samples from a very large list, [16], [10], [1], [5], [12], [4], [8]). This makes our Problem (*) quite natural. On the other hand, it is transparent that Problem (*) involves linear operators on indefinite inner product spaces. In this setting the formulations become simpler, and the "invariant" part (i.e., that independent from the chosen symmetries) can be pointed out. Of course, the usual difficulties of the "indefinite" case (e.g., the lack of an adequate substitute for the square root of "positive" operators) will show up.

In this paper we adapt the methods of [3] for giving a solution for Problem (*) in the Pontryagin case and when $\tilde{\kappa}$ has the least admissible value (see Theorem 5.3). Let us note that the existence problem for $\tilde{\kappa}$ bigger than the least admissible value can be easily deduced from there, but the description of all solutions involves (as suggested by Section 2) some new parameters and, on the other hand, some parameters may be unbounded.

For proving the main result, we need several facts which are presented in Sections 2–4. We begin by recalling the necessary Krein space terminology, and with

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